Mathieu Moonshine and Heterotic String Compactifications

Timm Wrase

Texas A&M

April 30, 2014

Based on: M. Cheng, X. Dong, J. Duncan, S. Harrison, S. Kachru, TW to appear
N. Paquette, TW to appear
TW 1402.2973
M. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, TW 1306.4981
Outline

• Introduction to moonshine

• Mathieu Moonshine and string compactifications

• Physical implications

• A new moonshine phenomena
Finite simple groups

There are 18 infinite families, e.g.

• Alternating group of n elements $A_n$
  e.g. $A_3$: $(123) \leftrightarrow (231) \leftrightarrow (312)$

• Cyclic groups of prime order $C_p$
  e.g.: $C_p = \mathbb{Z}_p = \langle e^{2\pi i/p} \rangle$
Finite simple groups

There are also 26 so called sporadic groups that do not come in infinite families:
Finite simple groups

There are also 26 so called sporadic groups that do not come in infinite families:

\[
|\text{Monster}| \approx 8 \times 10^{53}
\]
Finite simple groups

There are also 26 so called **sporadic groups** that do not come in infinite families:

- **Conway group** \( \approx 4 \times 10^{18} \)
- Largest Mathieu group \( \approx 2 \times 10^9 \)
- \( |M_{22}| = 443,520 \)
Modular Forms

Modular function of weight $k$

$$ f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) $$
Modular Forms

Modular function of weight $k$

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

Jacobi form of weight $k$ and index $m$

$$f\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right) = (c\tau + d)^k e^{\frac{2\pi imcz^2}{c\tau + d}} f(\tau, z)$$

$$f(\tau, z + \lambda\tau + \mu) = e^{-2\pi im(\lambda^2\tau + \lambda z)} f(\tau, z), \quad \lambda, \mu \in \mathbb{Z}$$
Modular Forms

Modular function of weight $k$

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

Jacobi form of weight $k$ and index $m$

$$f\left(\frac{a\tau+b z}{c\tau+d}, \frac{z}{c\tau+d}\right) = (c\tau+d)^k \exp\left[\frac{2\pi i m c z^2}{d\tau + d}\right] f(\tau, z)$$

$$f(\tau, z + \lambda \tau + \mu) = e^{-2\pi i m (\lambda^2 \tau + \lambda z)} f(\tau, z), \quad \lambda, \mu \in \mathbb{Z}$$

Can Fourier expand

$$f(\tau, z) = f(q = \exp[2\pi i \tau], y = \exp[2\pi i z]) = \sum_{n \geq 0} \sum_{r \leq 4mn} c(n, r) q^n y^r$$
Monstrous Moonshine

• The irreducible representations of the Monster group have dimensions 1, 196 883, 21 296 876, ...

• The J-function, that appears in many places in string theory, enjoys the expansion

\[ J(q) = \frac{1}{q} + 196884 q + 21493760 q^2 + \ldots \]
Monstrous Moonshine

• The irreducible representations of the Monster group have dimensions $1$, $196\,883$, $21\,296\,876$, ...

• The $J$-function, that appears in many places in string theory, enjoys the expansion

$$J(q) = \frac{1}{q} + 196884 q + 21493760 q^2 + ...$$

• as observed by John McKay
Monstrous Moonshine

moon·shine
ˈmʊnˌʃaɪn/ 🔊
noun informal
noun: moonshine

1. foolish talk or ideas.
Monstrous Moonshine

moon·shine
ˈmoonˌSHīn/  ≈
noun informal
noun: moonshine

1. foolish talk or ideas.

2. NORTH AMERICAN
illicitly distilled or smuggled liquor.
Monstrous Moonshine

This surprising connection can be explained by string theory:

- The (left-moving) bosonic string compactified on a $\mathbb{Z}_2$ orbifold of $\mathbb{R}^{24}/\Lambda$ with $\Lambda$ the Leech lattice has as its 1-loop partition function the $J(q)$-function

$$Z(q) = \text{Tr}_H q^{L_0 - \frac{c}{24}} = J(q) = \frac{1}{q} + 196884 q + 21493760 q^2 + ...$$
Monstrous Moonshine

This surprising connection can be explained by string theory:

- The (left-moving) bosonic string compactified on a $\mathbb{Z}_2$ orbifold of $\mathbb{R}^{24}/\Lambda$ with $\Lambda$ the Leech lattice has as its 1-loop partition function the $J(q)$-function

$$Z(q) = \text{Tr}_H q^{L_0 - c/24} = J(q) = \frac{1}{q} + 196884 q + 21493760 q^2 + \ldots$$

- Tachyon of the bosonic string
- No massless $q^0$ states
- Supermassive string states
Monstrous Moonshine

This surprising connection can be explained by string theory:

• The (left-moving) bosonic string compactified on a $\mathbb{Z}_2$ orbifold of $\mathbb{R}^{24}/\Lambda$ with $\Lambda$ the Leech lattice has as its 1-loop partition function the $J(q)$-function

$$Z(q) = \text{Tr}_H q^{L_0 - c/24} = J(q) = \frac{1}{q} + 196884 q + 21493760 q^2 + \ldots$$

• The symmetry group of the compactification space $\mathbb{R}^{24}/\Lambda/\mathbb{Z}_2$ is the Monster group.
Since we have a Virasoro algebra we can expand the $J(q)$-function in terms of Virasoro characters (traces of Verma modules)

$$
\text{ch}_{h=0}(q) = \frac{q^{-c/24}}{\prod_{n=2}^{\infty} (1 - q^n)}, \quad \text{ch}_h(q) = \frac{q^{h-c/24}}{\prod_{n=1}^{\infty} (1 - q^n)}
$$

$$
J(q) = \frac{1}{q} + 196884 q + 21493760 q^2 + \ldots
= \text{ch}_0(q) + \textcolor{blue}{196883 \text{ch}_2(q)} + \textcolor{red}{21296876 \text{ch}_3(q)} + \ldots
$$
Monstrous Moonshine

Group Theory
Representation Theory
finite (sporadic) groups

String Theory

Complex Analysis
Number Theory
(mock) modular forms, Jacobi forms

18
Monstrous Moonshine

Very interesting for mathematicians!
Monstrous Moonshine

Compactification of the bosonic string:

⇒ we have a tachyon (instability)

⇒ spacetime theory has no fermions

Additionally

• Only two spacetime dimensions are non-compact
Monstrous Moonshine

Compactification of the bosonic string:

\[ \implies \text{we have a tachyon (instability)} \]

\[ \implies \text{spacetime theory has no fermions} \]

Additionally

- Only two spacetime dimensions are non-compact

Not so interesting for physicists!
Mathieu Moonshine

• In 2010 Eguchi, Ooguri and Tachikawa discovered a new moonshine phenomenon that connects K3 to the largest Mathieu group $M_{24}$

Eguchi, Ooguri, Tachikawa 1004.0956

• They considered a $N=(4,4)$ SCFT with K3 target and calculate an index that is called elliptic genus
Mathieu Moonshine

Chemical potential for $U(1)$ in left-moving $N=2$ theory

$$Z_{\text{elliptic}}(q, y) = \text{Tr}_{\text{RR}} \left( (-1)^{F_L} q^{L_0 - \frac{c}{24}} y^{J_0} (-1)^{F_R} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right)$$

Witten index: No dependence on $\bar{q}$
Mathieu Moonshine

\[ Z_{\text{elliptic}}^{K3}(q, y) = 8 \left( \frac{\theta_2^2(q, y)}{\theta_2^2(q, 1)} + \frac{\theta_3^2(q, y)}{\theta_3^2(q, 1)} + \frac{\theta_4^2(q, y)}{\theta_4^2(q, 1)} \right) \]


We have N=(4,4) world sheet supersymmetry

\[ \Rightarrow \text{expand in N=4 Virasoro characters} \]
Mathieu Moonshine

\[ Z_{\text{elliptic}}^{K3}(q, y) = 8 \left( \frac{\theta_2(q, y)^2}{\theta_2(q, 1)^2} + \frac{\theta_3(q, y)^2}{\theta_3(q, 1)^2} + \frac{\theta_4(q, y)^2}{\theta_4(q, 1)^2} \right) \]


We have \( N=(4,4) \) world sheet supersymmetry

\[ \Rightarrow \text{expand in } N=4 \text{ Virasoro characters} \]

\( N=4 \) Virasoro characters are defined by the trace over the highest weight state and all its descendants

\[ \text{ch}_{h,l}(q, y) = \text{Tr} \left( q^{L_0 - \frac{c}{24}} y^{J_0} \right) \]

For the case \( h=c/24 \) there are short BPS multiplets
Mathieu Moonshine

\[ \mathcal{Z}^{K3}_{\text{elliptic}} (q, y) = 8 \left( \frac{\theta_2(q, y)^2}{\theta_2(q, 1)^2} \right) + \left( \frac{\theta_3(q, y)^2}{\theta_3(q, 1)^2} \right) + \left( \frac{\theta_4(q, y)^2}{\theta_4(q, 1)^2} \right) \]


We have N=(4,4) world sheet supersymmetry

\[ \implies \text{expand in N=4 Virasoro characters} \]

\[ \mathcal{Z}^{K3}_{\text{elliptic}} = 24 \text{ch}^{\text{short}}_{h=\frac{1}{4}, l=0} - 2 \text{ch}^{\text{long}}_{h=\frac{1}{4}, l=\frac{1}{2}} + \sum_{n=1}^{\infty} A_n \text{ch}^{\text{long}}_{h=\frac{1}{4}+n, l=\frac{1}{2}} \]

\[ A_n = \{90, 462, 1440, \ldots \} \]
Mathieu Moonshine

\[ Z^{K3}_{\text{elliptic}} (q, y) = 8 \left( \frac{\theta_2(q, y)^2}{\theta_2(q,1)^2} + \frac{\theta_3(q, y)^2}{\theta_3(q,1)^2} + \frac{\theta_4(q, y)^2}{\theta_4(q,1)^2} \right) \]


We have N=(4,4) world sheet supersymmetry

⇒ expand in N=4 Virasoro characters

\[ Z^{K3}_{\text{elliptic}} = 24 \left( \text{ch}^{\text{short}}_{h=\frac{1}{4}, l=0} - 2 \text{ch}^{\text{long}}_{h=\frac{1}{4}, l=\frac{1}{2}} + \sum_{n=1}^{\infty} A_n \text{ch}^{\text{long}}_{h=\frac{1}{4}+n, l=\frac{1}{2}} \right) \]

T. Eguchi, K. Hikami  0904.0911

\[ A_n = \{ 45 + 45, 231 + 231, 770 + 770, \ldots \} \]

T. Eguchi, H. Ooguri, Y. Tachikawa  1004.0956

Dimensions of Irreps of $M_{24}$
Mathieu Moonshine

Does this imply a connection between $M_{24}$ and K3?

• The geometric symmetries of K3 are contained in $M_{23} \subset M_{24}$

Mukai, Kondo 1988, 1998
Mathieu Moonshine

Does this imply a connection between $M_{24}$ and K3?

• The geometric symmetries of K3 are contained in $M_{23} \subset M_{24}$

Mukai, Kondo 1988, 1998

• The symmetry groups of $N=(4,4)$ SCFT with K3 target are never $M_{24}$ and for some points in moduli space do not even fit into $M_{24}$

Gaberdiel, Hohenegger, Volpato 1106.4315
Mathieu Moonshine

Does this imply a connection between $M_{24}$ and K3?

• The geometric symmetries of K3 are contained in $M_{23} \subset M_{24}$

  Mukai, Kondo 1988, 1998

• The symmetry groups of $N=(4,4)$ SCFT with K3 target are never $M_{24}$ and for some points in moduli space do not even fit into $M_{24}$

  Gaberdiel, Hohenegger, Volpato 1106.4315

• However, all coefficients $Z_{\text{elliptic}}^{K3}$ are positive sums of dimensions of $M_{24}$

  Gannon 1211.5531
Mathieu Moonshine

• K3 has played a central role in string compactifications and string dualities

• What are implications we can derive from Mathieu moonshine for string compactifications?

• Has the elliptic genus of K3 already appeared in the string theory literature?
Mathieu Moonshine

• K3 has played a central role in string compactifications and string dualities

• What are implications we can derive from Mathieu moonshine for string compactifications?

• Has the elliptic genus of K3 already appeared in the string theory literature?

YES!
Heterotic String Theory

• Consider the heterotic $E_8 \times E_8$ string theory compactified on $K3 \times T^2$

• We need to embed 24 instantons into $E_8 \times E_8 \rightarrow (12+n, 12-n)$ for $n = 0, 1, \ldots, 12$ to satisfy the Bianchi identity for $H_3$
Heterotic String Theory

• Consider the heterotic $E_8 \times E_8$ string theory compactified on $K3 \times T^2$

• We need to embed 24 instantons into $E_8 \times E_8 \rightarrow (12+n, 12-n)$ $n = 0, 1, ..., 12$ to satisfy the Bianchi identity for $H_3$

• The resulting four dimensional theories preserves $N=2$ spacetime supersymmetry

• The 1-loop corrections to the prepotential are related to the new supersymmetric index $Z_{\text{new}}$

\[
h(S,T,U) = h^{\text{tree}} + h^{1-\text{loop}} + O(e^{-2\pi i S})
\]

Dixon, Kaplunovsky, Louis, de Wit, Lüst, Stieberger, Antoniadis, Narain, Taylor, Gava, Kiritsis, Kounnas, Harvey, Moore, ....
The new supersymmetric index is defined as

\[ Z_{\text{new}} = \text{Tr}_R \left( \bar{J}_0 (-1)^{\bar{J}_0} q^{L_0 - \frac{c}{24}} \bar{q}^{L_0 - \frac{c}{24}} \right) \]

The trace is over our internal (22,9) conformal field theory for the heterotic $E_8 \times E_8$ string theory compactified on $K3 \times T^2$
The new supersymmetric index is defined as

\[ Z_{\text{new}} = \text{Tr}_R \left( \bar{J}_0 (-1)^F J_0 q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right) \]

The trace is over our internal (22,9) conformal field theory for the heterotic $E_8 \times E_8$ string theory compactified on $K3 \times T^2$.

We have a right moving $N=2$ SCFT from the $T^2$ and we denote its $U(1)$ generator $\bar{J}^{(1)}$.

For the $K3$ we have an $N=4$ SCFT with a level one $SU(2)$. We define $\bar{J}^{(2)} = 2 \bar{J}^3$ where $\bar{J}^3$ is the $SU(2)$ Cartan current.

Then $\bar{J} = \bar{J}^{(1)} + \bar{J}^{(2)}$.
Heterotic String Theory

- The new supersymmetric index is defined as

\[
Z_{\text{new}} = \text{Tr}_R \left( (\bar{J}_0^{(1)} + \bar{J}_0^{(2)}) (-1)^{\bar{J}_0^{(1)} + \bar{J}_0^{(2)}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} \right)
\]

\[
= \text{Tr}_R \bar{J}_0^{(1)} (-1)^{\bar{J}_0^{(1)} + \bar{J}_0^{(2)}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}
\]

\[
+ \text{Tr}_R (-1)^{\bar{J}_0^{(1)}} q^{L_0 - \frac{c}{24}} \bar{J}_0^{(2)} (-1)^{\bar{J}_0^{(2)}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}
\]
Heterotic String Theory

- The new supersymmetric index is defined as

\[
Z_{\text{new}} = \text{Tr}_R \left( (\bar{J}_0^{(1)} + \bar{J}_0^{(2)}) (-1)^{J_0^{(1)}+J_0^{(2)}} q^{L_0 - \frac{c}{24}} \bar{q}^{L_0 - \frac{\bar{c}}{24}} \right)
\]

\[
= \text{Tr}_R \bar{J}_0^{(1)} (-1)^{J_0^{(1)}+J_0^{(2)}} q^{L_0 - \frac{c}{24}} \bar{q}^{L_0 - \frac{\bar{c}}{24}}
\]

\[
+ \text{Tr}_R (-1)^{J_0^{(1)}} q^{L_0 - \frac{c}{24}} \bar{J}_0^{(2)} (-1)^{J_0^{(2)}} \bar{q}^{L_0 - \frac{\bar{c}}{24}}
\]

For SU(2) representations eigenvalues of \( J^{(2)} \) come in opposite pairs

\[
\text{Tr}_{K3} \bar{J}_0^{(2)} (-1)^{J_0^{(2)}} \bar{q}^{L_0 - \frac{\bar{c}}{24}}
\]

\[
\approx \sum_{n \in \mathbb{Z}} (n (-1)^n - n (-1)^{-n}) [...] = 0
\]
Heterotic String Theory

- The new supersymmetric index is defined as

\[
Z_{\text{new}} = \text{Tr}_R \left( (\bar{J}_0^{(1)} + \bar{J}_0^{(2)}) (-1)^{\bar{J}_0^{(1)} + \bar{J}_0^{(2)}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right) 
\]

\[
= \text{Tr}_R \bar{J}_0^{(1)} (-1)^{\bar{J}_0^{(1)} + \bar{J}_0^{(2)}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} 
\]

\[
+ \text{Tr}_R (-1)^{\bar{J}_0^{(1)}} q^{L_0 - \frac{c}{24}} \bar{J}_0^{(2)} (-1)^{\bar{J}_0^{(2)}} q^{\bar{L}_0 - \frac{\bar{c}}{24}} 
\]
Heterotic String Theory

The new supersymmetric index is defined as

\[ Z_{\text{new}} = \text{Tr}_R \left( (\bar{J}^{(1)}_0 + \bar{J}^{(2)}_0) (-1)^{J^{(1)}_0 + J^{(2)}_0} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right) \]

\[ = \text{Tr}_R \bar{J}^{(1)}_0 (-1)^{J^{(1)}_0 + J^{(2)}_0} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \]

\[ + \text{Tr}_R (-1)^{J^{(1)}_0} q^{L_0 - \frac{c}{24}} \bar{J}^{(2)}_0 (-1)^{J^{(2)}_0} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \]

\[ = \text{Tr}_R q^{L_0 - \frac{c}{24}} \bar{J}^{(1)}_0 (-1)^{J^{(1)}_0 + J^{(2)}_0} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \]

Now we calculate it for the standard embedding SU(2) ⊂ E_8 for a compactification on K3 × T^2
Heterotic String Theory

For K3 x T² compactifications we have for the standard embedding that preserves N=(4,4)

\[ Z_{\text{new}} = \text{Tr}_R q^{\frac{L_0}{24} \tilde{J}_0^{(1)}} (-1)^{\tilde{J}_0^{(1)} + \tilde{J}_0^{(2)}} \frac{L_0}{q^{\frac{c}{24}}} \]

Compare to

\[ Z_{\text{elliptic}} (q, y) = \text{Tr}_{\text{RR}} \left( (-1)^{F_L} q^{\frac{L_0}{24}} y^{J_0} (-1)^{F_R} \frac{L_0}{q^{\frac{c}{24}}} \right) \]

We will get contributions from \( Z^{\text{K3 elliptic}}_{\text{elliptic}} (q, y) \) at different y-values
For K3 x T^2 compactifications we have for the standard embedding that preserves N=(4,4)

\[ Z_{\text{new}}(q; T, U) = \frac{i}{2} \Theta_{\Gamma_{2,2}}(T, U) \frac{E_4(q)}{\eta(q)^4} \left[ \left( \frac{\theta_2(q)}{\eta(q)} \right)^6 Z_{\text{elliptic}}^{K3}(q, -1) \right. \]

\[ + \left. \left( \frac{\theta_3(q)}{\eta(q)} \right)^6 q^{\frac{1}{4}} Z_{\text{elliptic}}^{K3}(q, -q^{\frac{1}{2}}) - \left( \frac{\theta_4(q)}{\eta(q)} \right)^6 q^{\frac{1}{4}} Z_{\text{elliptic}}^{K3}(q, q^{\frac{1}{2}}) \right] \]

\[ T \text{ is the complexified Kähler modulus, } U \text{ the complex structure modulus of the } T^2 \]

Harvey, Moore hep-th/9510182
Heterotic String Theory

For K3 x T² compactifications we have for the standard embedding that preserves N=(4,4)

\[
Z_{\text{new}}(q; T, U) = \frac{i}{2} \left( \frac{\Theta_{\Gamma_2}(T, U)}{\eta(q)^4} \right) \frac{E_4(q)}{\eta(q)^8} \left[ \left( \frac{\theta_2(q)}{\eta(q)} \right)^6 \right. \\
\left. \left( \frac{\theta_3(q)}{\eta(q)} \right)^{\frac{1}{4}} \right]
\]

\[
+ \left( \frac{\theta_3(q)}{\eta(q)} \right)^6 q^{\frac{1}{4}} \left[ \frac{Z_{\text{elliptic}}^{K3}(q, -q^{\frac{1}{2}})}{\eta(q)^6} \right] - \left( \frac{\theta_4(q)}{\eta(q)} \right)^6 q^{\frac{1}{4}} \left[ \frac{Z_{\text{elliptic}}^{K3}(q, q^{\frac{1}{2}})}{\eta(q)^6} \right]
\]

So in particular the [...] part has an “SO(12)xM_{24}”-expansion: exactly the same M_{24} as in Mathieu Moonshine due to N=(4,4)
Heterotic String Theory

For K3 x T^2 compactifications we have for the standard embedding that preserves N=(4,4)

\[
Z_{new}(q; T, U) = \frac{i}{2} \Theta_{\Gamma_{2,2}}(T, U) \frac{E_4(q)}{\eta(q)^4} \eta(q)^8 \left[ + 24 \left( \frac{\theta_2(q)}{\eta(q)} \right)^6 \text{ch}_{h=\frac{1}{4}, l=0}^{\text{short}} (q, -1) + \left( \frac{\theta_3(q)}{\eta(q)} \right)^6 q^{\frac{1}{4}} \text{ch}_{h=\frac{1}{4}, l=0}^{\text{short}} (q, -q^{\frac{1}{2}}) - \left( \frac{\theta_4(q)}{\eta(q)} \right)^6 q^{\frac{1}{4}} \text{ch}_{h=\frac{1}{4}, l=0}^{\text{short}} (q, q^{\frac{1}{2}}) \right] \\
- 2 \left( \frac{\theta_2(q)}{\eta(q)} \right)^6 \text{ch}_{h=\frac{1}{4}, l=\frac{1}{2}}^{\text{short}} (q, -1) + \left( \frac{\theta_3(q)}{\eta(q)} \right)^6 q^{\frac{1}{4}} \text{ch}_{h=\frac{1}{4}, l=\frac{1}{2}}^{\text{short}} (q, -q^{\frac{1}{2}}) - \left( \frac{\theta_4(q)}{\eta(q)} \right)^6 q^{\frac{1}{4}} \text{ch}_{h=\frac{1}{4}, l=\frac{1}{2}}^{\text{short}} (q, q^{\frac{1}{2}}) \right] \\
+ \left( \frac{\theta_2(q)}{\eta(q)} \right)^6 \text{ch}_{h=\frac{1}{4}, l=\frac{1}{2}}^{\text{long}} (q, -1) + \left( \frac{\theta_3(q)}{\eta(q)} \right)^6 q^{\frac{1}{4}} \text{ch}_{h=\frac{1}{4}, l=\frac{1}{2}}^{\text{long}} (q, -q^{\frac{1}{2}}) - \left( \frac{\theta_4(q)}{\eta(q)} \right)^6 q^{\frac{1}{4}} \text{ch}_{h=\frac{1}{4}, l=\frac{1}{2}}^{\text{long}} (q, q^{\frac{1}{2}}) \right] \sum_{n=1}^{\infty} A_n q^n \right] \\
A_n = \{ 45 + 45, 231 + 231, 770 + 770, ... \}
\]
Heterotic String Theory

For $K3 \times T^2$ compactifications we have for the standard $(24,0)$ instanton embedding

$$Z_{\text{new}}(q;T,U) = \frac{i}{2} \frac{\Theta_{2,2}(T,U)}{\eta(q)^4} \frac{E_4(q)}{\eta(q)^8} \frac{E_6(q)}{\eta(q)^{12}} = \frac{i}{2} \frac{\Theta_{2,2}(T,U)E_4(q)E_6(q)}{\eta(q)^{24}}$$

Harvey, Moore  hep-th/9510182

So in particular the $E_6(q)$ has an “$\text{SO}(12)\times M_{24}$”-expansion
Heterotic String Theory

For K3 x T\(^2\) compactifications we have for the standard embedding that preserves N=(4,4)

\[
\text{Take away message:}
\]

\[
Z_{\text{new}} \text{ depends on } T \text{ and } U \text{ and is connected to } Z_{\text{elliptic}} \text{ and therefore to } M_{24}
\]

\(T\) is the complexified Kähler modulus, \(U\) the complex structure modulus of the \(T^2\)
Heterotic String Theory

The 1-loop correction to the prepotential is roughly determined by

\[ \Delta(T, U) = \int \frac{d^2 \tau}{\tau_2} Z_{\text{new}}(q = e^{2\pi i \tau}; T, U) \left( Q^2 - \frac{1}{8\pi^2} \right) \]

and knows about \( M_{24} \) since \( Z_{\text{new}} \) does

M. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, TW 1306.4981
Heterotic String Theory

The 1-loop correction to the prepotential is roughly determined by

$$\Delta(T,U) = \int \frac{d^2 \tau}{\tau_2} Z_{\text{new}}(q = e^{2\pi i \tau}; T, U) \left( Q^2 - \frac{1}{8\pi^2} \right)$$

and knows about $M_{24}$ since $Z_{\text{new}}$ does

M. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, TW 1306.4981

The modular invariance of $\tau_2 Z_{\text{new}}(Q^2 - \frac{1}{8\pi^2})$ actually tells us that there is a unique solution. So for all instanton embeddings $(12+n, 12-n)$ the answer is the same.

Kiritsis, Kounnas, Petropoulos, Rizos hep-th/9608034
Henningson, Moore hep-th/9608145
Heterotic String Theory

The 1-loop correction to the prepotential is roughly determined by

$$\Delta(T,U) = \int \frac{d^2 \tau}{\tau_2} Z_{\text{new}}(q = e^{\frac{2\pi i}{\tau}}; T, U) \left( Q^2 - \frac{1}{8\pi^2} \right)$$

and knows about $M_{24}$ since $Z_{\text{new}}$ does

M. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, TW 1306.4981

We have to solve the following second order differential equation

Harvey, Moore hep-th/9510182

$$- \Re \left\{ \partial_T \partial_U h^{1\text{-loop}} + \frac{1}{T_1 U_1} (1 - T_1 \partial_T - U_1 \partial_U) h^{1\text{-loop}} \right\} - \frac{1}{\pi} \Re(\log[J(iT) - J(iU)])$$

$$= \frac{1}{2\pi} \int \frac{d^2 \tau}{\tau_2} \left( -i Z_{\text{new}}(q; T, U) \cdot (Q_{E_8}^2 - \frac{1}{8\pi^2}) - b(E_8) \right) + \frac{b(E_8)}{2\pi} \left( \log[2T_1 U_1] + 4 \Re(\log[\eta(iT)\eta(iU)]) \right)$$
Heterotic String Theory

The solution is given by

\[ h^{1\text{-}\text{loop}} = -\frac{1}{3} U^3 + C + \sum_{k,l} c(kl) \text{Li}_3(e^{2\pi i(kT+lU)}) \]

where the polylogarithm is given by

\[ \text{Li}_3(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3} \]
Heterotic String Theory

The solution is given by

\[ h^{1-loop} = -\frac{1}{3} U^3 + C + \sum_{k,l} c(kl) \text{Li}_3 \left( e^{2\pi i (kT + lU)} \right) \]

where the polylogarithm is given by \( \text{Li}_3(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3} \)

and the expansion coefficients are the same as in our index (they go along for the ride when integrating)

\[ Z_{\text{new}}(q;T,U) = \frac{i}{2} \Theta_{\Gamma_{2,2}} (T,U) \left( \sum_{m \geq -1} c(m) q^m \right) \]
Heterotic String Theory

The solution is given by

\[ h^{1\text{-loop}} = -\frac{1}{3} U^3 + C + \sum_{k,l} c(kl) Li_3(e^{2\pi i(kT+lU)}) \]

where the polylogarithm is given by \( Li_3(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3} \)

and the expansion coefficients are the same as in our index (they go along for the ride when integrating)

\[ Z_{\text{new}}(q; T, U) = \frac{i}{2} \Theta_{\Gamma_{2,2}}(T, U) \left( \sum_{m \geq -1} c(m) q^m \right) \]
Type IIA on CY_3

String duality

Heterotic string on K3 x T^2 with instanton embedding (12+n,12-n)

Type IIA string theory on elliptic fibrations over F_n (Hirzebruch surface)
Type IIA on CY$_3$

**String duality**

Heterotic string on K3 x T$^2$ with instanton embedding (12+n,12-n)

Type IIA string theory on elliptic fibrations over F$_n$ (Hirzebruch surface)

\[ \text{dilaton } S \leftrightarrow \text{Size of base } S^2 \]
Type IIA on CY$_3$

Type IIA string theory on elliptic fibrations over $F_n$:

• Prepotential receives instanton corrections

• These are determined by the Gromow-Witten invariants $\approx$ curve counting ($S^2$, $T^2$, $\ldots$)
Type IIA on CY$_3$

Type IIA string theory on elliptic fibrations over $F_n$:

- Prepotential receives instanton corrections
- These are determined by the Gromow-Witten invariants $\approx$ curve counting ($S^2$, $T^2$, ...)

$$h(S,T,U) = -STU - \frac{1}{3} U^3 + C + \sum_{k,l} c(kl) Li_3(e^{2\pi i(kT+lU)}) + O(e^{-2\pi iS})$$

M. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, TW 1306.4981

M. Alim, E. Scheidegger 1205.1784
A. Klemm, J. Manschot, T. Wotschke 1205.1795

Dimensions of $M_{24}$
(appearing in a spacetime quantity)
Type IIB on CY$_3$

String duality

Type IIA string theory on elliptic fibrations over $F_n$ (Hirzebruch surface)

Type IIB string theory on mirror manifold
Type IIB on CY\(_3\)

**String duality**

Type IIA string theory on elliptic fibrations over 
\(F_n\) (Hirzebruch surface)

Type IIB string theory on mirror manifold

---

CY\(_3\) manifold \(X_n\) \(\leftrightarrow\) CY\(_3\) manifold \(Y_n\)

Gromov-Witten invariants \(\leftrightarrow\) Periods of the holomorphic 3-form \(\Omega\)
Cool new math connections!

**Group Theory**
- Representation Theory
- finite (sporadic) groups

**Complex Analysis**
- Number Theory
- (mock) modular forms
- Jacobi forms

**String Theory**
- periods of Calabi-Yau manifolds
- Gromov-Witten invariants
- elliptic genus
- ...

**(Algebraic) Geometry**
- finite (sporadic) groups
- (mock) modular forms
- Jacobi forms

**Cool new math connections!**
Moonshine and physics

“I have a sneaking hope, a hope unsupported by any facts or any evidence, that sometime in the twenty-first century physicists will stumble upon the Monster group, built in some unsuspected way into the structure of the Universe.”

– Freeman Dyson (1983)
Moonshine and physics

For the K3xT^2 compactifications, the 1-loop prepotential controls the 1-loop corrections to the gauge couplings in the N=2 spacetime theory
Moonshine and physics

For the K3xT^2 compactifications, the 1-loop prepotential controls the 1-loop corrections to the gauge couplings in the N=2 spacetime theory.

For **four dimensional N=1 models** obtained from orbifold compactifications of the heterotic E_8 x E_8 string theory:

\[
f_\alpha(S, T, U) = S + f_\alpha^{1\text{-loop}}(T, U) + O(e^{-2\pi i S})
\]
Moonshine and physics

For the K3xT^2 compactifications, the 1-loop prepotential controls the 1-loop corrections to the gauge couplings in the N=2 spacetime theory

For four dimensional N=1 models obtained from orbifold compactifications of the heterotic E_8 x E_8 string theory:

\[ f_\alpha(S, T, U) = S + f^{1\text{-loop}}_\alpha(T, U) + O(e^{-2\pi i S}) \]

The (bulk) moduli dependent 1-loop correction to the gauge kinetic function arises only from N=2 subsectors!

Moonshine and physics

Example $\mathbb{T}^6/\mathbb{Z}_{6-II} = \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2/\mathbb{Z}_{6-II}$:

$$\mathbb{Z}_{6-II} = \langle g \rangle, \quad g : (z_1, z_2, z_3) \rightarrow (e^{\pi i/3} z_1, e^{2\pi i/3} z_2, -z_3)$$
Moonshine and physics

Example $T^6/Z_{6-II} = T^2 \times T^2 \times T^2/Z_{6-II}$:

$Z_{6-II} = \langle g \rangle, \quad g : (z_1, z_2, z_3) \rightarrow (e^{\pi i/3} z_1, e^{2\pi i/3} z_2, - z_3)$

has two N=2 subsector

$Z_3 = \{1, g^2, g^4\}, \quad g^2 : (z_1, z_2, z_3) \rightarrow (e^{2\pi i/3} z_1, e^{4\pi i/3} z_2, z_3)$

$Z_2 = \{1, g^3\}, \quad g^3 : (z_1, z_2, z_3) \rightarrow (- z_1, z_2, - z_3)$
Moonshine and physics

Example $T^6/Z_{6-II} = T^2 \times T^2 \times T^2/Z_{6-II}$:

$$Z_{6-II} = \langle g \rangle, \quad g : (z_1, z_2, z_3) \rightarrow (e^{\pi i/3} z_1, e^{2\pi i/3} z_2, -z_3)$$

has two N=2 subsector

$$Z_3 = \{1, g^2, g^4\}, \quad g^2 : (z_1, z_2, z_3) \rightarrow (e^{2\pi i/3} z_1, e^{4\pi i/3} z_2, z_3)$$

$$Z_2 = \{1, g^3\}, \quad g^3 : (z_1, z_2, z_3) \rightarrow (-z_1, z_2, -z_3)$$

For which the internal space is $T^4/Z_3 \times T^2$ or $T^4/Z_2 \times T^2$ respectively and therefore an orbifold limit of $T^2 \times K3$. 
Moonshine and physics

N=2 sectors lead to 1-loop corrections

\[ f_{\alpha}^{1-\text{loop}}(T,U) = \sum_{i=1,2,3} \frac{|G_i|}{|G|} \left[ - \frac{1}{2} \partial_{T_i} \partial_{U_i} h_i^{1-\text{loop}}(T_i,U_i) \right. \]

\[ \left. - \frac{1}{8\pi^2} \log[ J(iT_i) - J(iU_i) - \frac{b_{\alpha,i}^{N=2}}{4\pi^2} (\log[ \eta(iT_i)\eta(iU_i)] ) ] \right] \]
Moonshine and physics

N=2 sectors lead to 1-loop corrections

\[ f_{\alpha}^{1\text{-loop}}(T,U) = \sum_{i=1,2,3} \frac{|G_i|}{|G|} \left[ -\frac{1}{2} \partial_{T_i} \partial_{U_i} h_i^{1\text{-loop}}(T_i,U_i) \right. \]
\[ \left. - \frac{1}{8\pi^2} \log[ J(iT_i) - J(iU_i) - \frac{b_{\alpha,i}^{N=2}}{4\pi^2} (\log[ \eta(iT_i)\eta(iU_i)]) ] \right] \]

where the prepotential was calculated above

\[ h^{1\text{-loop}}(T,U) = -\frac{1}{3} U^3 + C + \sum_{k,l} c(kl) Li_3(e^{2\pi i(kT+lU)}) \]

Dimensions of $M_{24}$
Moonshine and physics

<table>
<thead>
<tr>
<th>Group $\mathbb{Z}_N$</th>
<th>Generator $\frac{1}{N}(\varphi_1, \varphi_2, \varphi_3)$</th>
<th>$\mathcal{N} = 2$ moduli</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{Z}_3$</td>
<td>$\frac{1}{3}(1, 1, 1)$</td>
<td>-</td>
</tr>
<tr>
<td>$\mathbb{Z}_4$</td>
<td>$\frac{1}{4}(1, 1, 2)$</td>
<td>$T_3, U_3$</td>
</tr>
<tr>
<td>$\mathbb{Z}_{6-I}$</td>
<td>$\frac{1}{6}(1, 1, 4)$</td>
<td>$T_3$</td>
</tr>
<tr>
<td>$\mathbb{Z}_{6-II}$</td>
<td>$\frac{1}{6}(1, 2, 3)$</td>
<td>$T_2, T_3, U_3$</td>
</tr>
<tr>
<td>$\mathbb{Z}_7$</td>
<td>$\frac{1}{7}(1, 2, 4)$</td>
<td>-</td>
</tr>
<tr>
<td>$\mathbb{Z}_{8-I}$</td>
<td>$\frac{1}{8}(1, 2, 5)$</td>
<td>$T_2$</td>
</tr>
<tr>
<td>$\mathbb{Z}_{8-II}$</td>
<td>$\frac{1}{8}(1, 3, 4)$</td>
<td>$T_3, U_3$</td>
</tr>
<tr>
<td>$\mathbb{Z}_{12-I}$</td>
<td>$\frac{1}{12}(1, 4, 7)$</td>
<td>$T_2$</td>
</tr>
<tr>
<td>$\mathbb{Z}_{12-II}$</td>
<td>$\frac{1}{12}(1, 5, 6)$</td>
<td>$T_3, U_3$</td>
</tr>
</tbody>
</table>

No $\mathcal{N}=2$ sectors
## Moonshine and physics

<table>
<thead>
<tr>
<th>( \mathbb{Z}_N \times \mathbb{Z}_M )</th>
<th>1(^{st}) generator ( \frac{1}{N}(\varphi_1, \varphi_2, \varphi_3) )</th>
<th>2(^{nd}) generator ( \frac{1}{M}(\dot{\varphi}_1, \dot{\varphi}_2, \dot{\varphi}_3) )</th>
<th>( \mathcal{N} = 2 ) moduli</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_2 )</td>
<td>( \frac{1}{2}(1,0,1) )</td>
<td>( \frac{1}{2}(0,1,1) )</td>
<td>( T_1, U_1, T_2, U_2, T_3, U_3 )</td>
</tr>
<tr>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_4 )</td>
<td>( \frac{1}{2}(1,0,1) )</td>
<td>( \frac{1}{4}(0,1,3) )</td>
<td>( T_1, U_1, T_2, T_3 )</td>
</tr>
<tr>
<td>( \mathbb{Z}_2 \times \mathbb{Z}_6 )</td>
<td>( \frac{1}{2}(1,0,1) )</td>
<td>( \frac{1}{6}(0,1,5) )</td>
<td>( T_1, U_1, T_2, T_3 )</td>
</tr>
<tr>
<td>( \mathbb{Z}_2 \times \mathbb{Z}'_6 )</td>
<td>( \frac{1}{2}(1,0,1) )</td>
<td>( \frac{1}{6}(1,1,4) )</td>
<td>( T_1, T_2, T_3 )</td>
</tr>
<tr>
<td>( \mathbb{Z}_3 \times \mathbb{Z}_3 )</td>
<td>( \frac{1}{3}(1,0,2) )</td>
<td>( \frac{1}{3}(0,1,2) )</td>
<td>( T_1, T_2, T_3 )</td>
</tr>
<tr>
<td>( \mathbb{Z}_3 \times \mathbb{Z}_6 )</td>
<td>( \frac{1}{3}(1,0,2) )</td>
<td>( \frac{1}{6}(0,1,5) )</td>
<td>( T_1, T_2, T_3 )</td>
</tr>
<tr>
<td>( \mathbb{Z}_4 \times \mathbb{Z}_4 )</td>
<td>( \frac{1}{4}(1,0,3) )</td>
<td>( \frac{1}{4}(0,1,3) )</td>
<td>( T_1, T_2, T_3 )</td>
</tr>
<tr>
<td>( \mathbb{Z}_6 \times \mathbb{Z}_6 )</td>
<td>( \frac{1}{6}(1,0,5) )</td>
<td>( \frac{1}{6}(0,1,5) )</td>
<td>( T_1, T_2, T_3 )</td>
</tr>
</tbody>
</table>
Moonshine and physics

Four dimensional $N=1$ models obtained from orbifold compactifications of the heterotic $E_8 \times E_8$ string theory receive universal 1-loop corrections to their gauge kinetic functions that are related to $M_{24}$

TW 1402.2973
Moonshine and physics

Four dimensional N=1 models obtained from orbifold compactifications of the heterotic $E_8 \times E_8$ string theory receive universal 1-loop corrections to their gauge kinetic functions that are related to $M_{24}$

For all $T^6/\mathbb{Z}_N$, $N \neq 3,7$, and all $T^6/\mathbb{Z}_N \times \mathbb{Z}_M$

$$f(S,T,U) \approx S + \partial_T \partial_U \sum_{k,l} c(kl) \text{Li}_3(e^{2\pi i (kT+lU)}) + \ldots + O(e^{-2\pi i S})$$

Dimensions of $M_{24}$
Moonshine and physics

N. Paquette, TW work in progress:

• The holomorphic 3-form $\Omega$ plays a role in flux compactifications

Gukov, Vafa, Witten hep-th/9906070
Giddings, Kachru, Polchinski hep-th/0105097

$$W = \int H \wedge \Omega + \ldots$$
Moonshine and physics

N. Paquette, TW work in progress:

• The holomorphic 3-form $\Omega$ plays a role in flux compactifications

  $W = \int H \wedge \Omega + ...$

  Flux vacua might have large symmetry groups: $|M_{24}| \approx 2 \times 10^9$

Gukov, Vafa, Witten hep-th/9906070
Giddings, Kachru, Polchinski hep-th/0105097
Moonshine and physics

N. Paquette, TW work in progress:

• The holomorphic 3-form $\Omega$ plays a role in flux compactifications

  \[ W = \int H \wedge \Omega + ... \]

  Gukov, Vafa, Witten hep-th/9906070
  Giddings, Kachru, Polchinski hep-th/0105097

• The Yukawa couplings in heterotic models are given by the third derivative of $\Omega$ with respect to the moduli

  \[ Y_{IJK} \approx \partial_I \partial_J \partial_K h(S, T, U) \]

  Hosono, Klemm, Theisen, Yau hep-th/9308122
New Moonshine phenomena

Consider eight (left-moving) bosons and fermions compactified on the orbifold $T^8/Z_2 = \mathbb{R}^8/\Lambda_{E8} / Z_2$

The partition function in the NS sector is

$$Z(q) = \frac{1}{\sqrt{q}} + 276\sqrt{q} + 2048q + 11202q^{\frac{3}{2}} + ...$$

Frenkel, Lepowsky, Meurman 1985
Duncan math/0502267

Sums of dimensions of Conway group
New Moonshine phenomena

Consider eight (left-moving) bosons and fermions compactified on the orbifold $T^8/Z_2 = \mathbb{R}^8/\Lambda_{E8} / Z_2$

Tensor left- and right movers together in an asymmetric orbifold where we act with one $Z_2$ on the left and with another one only on the right:

$$Z_{\text{partition}}(q, \bar{q}) = Z(q) Z(\bar{q})$$

The asymmetric orbifold preserves $N=(4,4)$ symmetry
New Moonshine phenomena

Asymmetric orbifold \( T^8/Z_2 \times Z_2 = \mathbb{R}^8/\Lambda_{E8}/Z_2 \times Z_2 \)

Calculate the elliptic genus and expand in N=4 characters:

\[
Z_{\text{elliptic}} = 21 \text{ch}^{\text{short}}_{h=\frac{1}{2},l=0} + \text{ch}^{\text{short}}_{h=\frac{1}{2},l=1} + 560 \text{ch}^{\text{long}}_{h=\frac{3}{2},l=\frac{1}{2}} + 8470 \text{ch}^{\text{long}}_{h=\frac{5}{2},l=\frac{1}{2}} + \ldots
\]

\[
+ 210 \text{ch}^{\text{long}}_{h=\frac{3}{2},l=1} + 4444 \text{ch}^{\text{long}}_{h=\frac{5}{2},l=1} + \ldots
\]

Two infinite series, coefficients unrelated to Conway
New Moonshine phenomena

Asymmetric orbifold $T^8/\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{R}^8/\Lambda_{E8}/\mathbb{Z}_2 \times \mathbb{Z}_2$

Calculate the elliptic genus and expand in N=4 characters:

$$ Z_{\text{elliptic}} = 21 \text{ch}^{\text{short}}_{h=\frac{1}{2}, l=0} + \text{ch}^{\text{short}}_{h=\frac{1}{2}, l=1} + 560 \text{ch}^{\text{long}}_{h=\frac{3}{2}, l=\frac{1}{2}} + 8470 \text{ch}^{\text{long}}_{h=\frac{5}{2}, l=\frac{1}{2}} + ... + 210 \text{ch}^{\text{long}}_{h=\frac{3}{2}, l=1} + 4444 \text{ch}^{\text{long}}_{h=\frac{5}{2}, l=1} + ... $$

All coefficients are dimensions of the Mathieu group $M_{22}$!

Checks all work and one can see $M_{22} \subset \text{Co}$ preserving N=4

M. Cheng, X. Dong, J. Duncan, S. Harrison, S. Kachru, TW to appear
Conclusion

• Mathieu Moonshine involves K3 that has played a crucial role in superstring compactifications and string dualities

• Certain CY$_3$ manifolds are now also implicated in Mathieu Moonshine

• This leads to a variety of intriguing implications for physically interesting string compactifications

• Much more to come!
Conclusion

• Mathieu Moonshine involves K3 that has played a crucial role in superstring compactifications and string dualities

• Certain CY\textsubscript{3} manifolds are now also implicated in Mathieu Moonshine

• This leads to a variety of intriguing implications for physically interesting string compactifications

• Much more to come!

THANK YOU!