Superstring Field Theory on the Small Hilbert Space

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Plan:

- Review of bosonic OSFT
- Deformation theory
- NS-sector of Superstring field theory
- R-sector
tree-level open (bosonic) SFT:

\[ \sim \text{decomposition of the moduli space of disks with punctures} \]

\[ B-V: \quad \partial \mathcal{V}_4 + (\mathcal{V}_3, \mathcal{V}_3) = 0 \]

structure
boundary operator
\rightarrow \text{glueing} \sim
tree-level open (bosonic) SFT:

decomposition of the moduli space of disks with punctures

\[ \langle \psi_2 | \square | \psi_4 \rangle \]

\[ \nabla V_4 + (V_3 \cdot V_3) = 0 \]

CFT: \[ \mathcal{H}_0, \ \omega(\psi_1, \psi_2), \ \omega(\psi_1, \psi_2 + \psi_3) \]

\[ \mathcal{H}_{\text{BRST}} \] \[ \text{BPZ inner product} \]

\[ \Rightarrow (S, S) = 0 \]

CFT defines a morphism between B-V algebras
Witten's OSFT: \[ S[\psi] = \omega(\psi, \varphi \psi) + \frac{g}{3!} \omega(\psi, \varphi \psi \times \varphi \psi) \]

\[ |\psi\rangle = g h(\psi) - 1 \]

\[ V_3 = \text{pt}. \]

\[ \hat{m} = \mathbb{Q} \otimes \mathbb{Q} \otimes \mathbb{Q} \ldots + m_2 \otimes \mathbb{Q} \otimes \mathbb{Q} \ldots \]

\[ \hat{m} : \mathcal{H}_0 \otimes \mathcal{H}_0 \otimes \mathcal{H}_0 \otimes \mathcal{H}_0 \ldots \rightarrow \mathcal{H}_0 \otimes \mathcal{H}_0 \otimes \mathcal{H}_0 \otimes \mathcal{H}_0 \otimes \mathcal{H}_0 \ldots \]

\[ \equiv T \mathcal{H}_0 \]

\[ (S, S) = 0 \iff \hat{m}^2 = [\hat{m}, \hat{m}] = 0 \quad \text{(plus cyclicity)} \]

\[ \Rightarrow \text{graded commutator} \]
\[
S[\psi] = \omega(\psi, \partial_0 \psi) + \frac{g}{3!} \omega(\psi, \psi \times \psi) + \omega(\psi, m_3(\psi, \psi, \psi)) + \cdots
\]

\[
\mathfrak{V}_3 = \mathfrak{p}^t + \mathfrak{V}_4 + \cdots
\]

\[\hat{m} = \eta \otimes 1 \otimes 1 \cdots + m_2 \otimes 1 \otimes 1 \cdots + m_3 \otimes 1 \otimes 1 \cdots \]

\[\left[ \hat{m} + \delta \hat{m}, \hat{m} + \delta \hat{m} \right] = \left[ \hat{m}, \hat{m} \right] + \left[ \hat{m}, \delta \hat{m} \right] + \cdots \quad \overset{\text{B-V}}{=} 0
\]

\[\Rightarrow \text{consistent def'n of OSFT is a cohomology problem}
\]

\[d_H = \left[ \hat{m}, \cdot \right] \quad \text{e.g. adding strips is a trivial, } d_H\text{-exact def'n}
\]

"Thm" (N. Moeller, I. S.): \(\text{coh}(d_H) \cong \text{coh}(Q_{\text{closed}})\)
NS-sector: I am not aware of an analogue of a B-V equation for super moduli space. Usually one just assumes a decomposition that is mapped by a SCFT (matter + b.c + B,γ) to a B-V equation for $\mathcal{S}[\phi,\bar{\phi}]$.

\[
\phi^{(-1)} \quad (X \phi^{(-1)}) \cong \text{integ'd vertex op.} \quad \mathcal{V} \text{ cyclicity}
\]

\[
\gamma = \eta e^\phi, \quad \beta = \partial \xi e^{-\phi}
\]

\[
V(z) = c \delta(y) V_m = ce^{-\phi} V_m
\]

\[
(\xi \in \mathcal{H}_{\text{large}}) \quad \text{Berkovits} \quad \text{SFT}
\]

\[
\text{Witten '86:} \quad \text{However, the 4-pt. fn diverges}
\]
Alternative: (Euler, Konopka, I.S. '13)

\[ X \rightarrow X + \frac{x}{X} + \frac{x}{X} \]

\( X(i) \rightarrow X = \int f(z) X(z) \) in local coord,

eg. \( f(z) = \frac{1}{z} \) \( \rightarrow X \leftrightarrow X_0 \)

in terms of maps:

\[ M_2(\cdot, \cdot) = \frac{4}{3} \left( X m_2(\cdot, \cdot) + m_2(X, \cdot, \cdot) + m_2(\cdot, X, \cdot) \right) \]
Problems:

i) doesn't reproduce the perturb. 4-pt. amplitude:

\[ \neq \]

ii) not gauge-invariant:

\[ M_2 \left( M_2 (A, B), C \right) + (-1)^{[AB]} M_2 (A, M_2 (B, C)) \neq 0 \]

geometrically,

\[ \neq 0 \quad \text{or} \quad \neq 0 \quad \text{(see also Sen, Witten '15)} \]
Both problems can be solved simultaneously by introducing a contact interaction $M_3$ s.t.

no bosonic moduli

$$M_2 \left( M_2(A,B), C \right) + (-1)^{\text{tr} A_1} M_2(A, M_2(B,C))$$

$$+ Q M_3(A,B,C) + M_3(QA,B,C) + \text{cycl.} = 0$$

Can solve for $M_3$ by noticing that $M_2$ is Q-exact in the large Hilbert space.
\[ M_2 = \{Q, \mu_2\} \, , \quad \xi = \int f(\xi) \, \mu(\xi) \]

\[ m_2(\cdot, \cdot) = \frac{4}{3} \left( \xi m_2(\cdot, \cdot) - m_2(\xi, \cdot) - (-1)^{1/3} m_2(\cdot, \xi, \cdot) \right) \]

then:

\[ M_3 = \frac{4}{2} \left( M_2 (\cdot, m_2 (\cdot, \cdot) + \text{perm.}) + \frac{1}{2} [Q, \mu_3] \right) \]

\( \mu_3 \) is a priori arbitrary but can be fixed by imposing that \( M_3 \) preserves the Small Hilbert Space.
\[ 0 = [\eta, M_3] \]

\[ = \frac{1}{3} \left[ (-1)^n m_2 (\cdot, m_2 (\cdot), \cdot) + m_2 (\xi m_2 (\cdot, \cdot), \cdot) \right] + \frac{1}{2} [\eta, [\eta, M_3]] \]

\[ = Q \left\{ \frac{1}{3} \left[ (-1)^n m_2 (\cdot, m_2 (\cdot), \cdot) + m_2 (\xi m_2 (\cdot, \cdot), \cdot) \right] + \frac{1}{2} [\eta, M_3] \right\} \]

\[ \equiv \frac{2}{3} m_3 = \sqrt{\text{m}_2 \text{ is associative}} \]

Now, since \[ [\eta, M_3] = [m_2, m_2] = 0 \]

\[ M_3 = \frac{1}{4} \left[ \xi m_3 (\cdot, \cdot, \cdot) - m_3 (\xi, \cdot, \cdot) + \text{perm.} \right] \]

guarantees that \[ [\eta, M_3] = 0 \].
Quartic vertex:

\[ \omega_L : \text{symplectic form in } \mathcal{H}_L \]

\[ M_2 = * \]
Recurrence relation: \[ M_{n+2} = \frac{1}{n+1} \sum_{k=0}^{n} [M_{k+1}, M_{n-k+2}] \]
\[ M_1 = \mathbb{0} \]

Let:
\[ \hat{M} = M_2 \otimes 1 \otimes 1 \ldots + M_3 \otimes 1 \otimes 1 \ldots + \ldots, \quad \hat{\mu} = M_2 \otimes 1 \otimes 1 + \ldots \]

then
\[ \hat{M}_3 = \frac{1}{2} \left( [\hat{M}_2, \hat{\mu}_2] + [\hat{\alpha}, \hat{M}_3] \right) = \frac{1}{2} [\hat{M}, \hat{\mu}]_3 \]
Generating functions:

Let

\[ \hat{M}(t) = \sum_{n=0}^{\infty} t^n \hat{M}_{n+1} \]

\[ \hat{m}(t) = \sum_{n=0}^{\infty} t^n \hat{m}_{n+2} \]

\[ \hat{\mu}(t) = \sum_{n=0}^{\infty} t^n \hat{\mu}_{n+2} \]

then:

\[ \frac{d}{dt} \hat{M}(t) = [\hat{M}(t), \hat{\mu}(t)] ; \quad \frac{d}{dt} \hat{m}(t) = [\hat{m}(t), \hat{\mu}(t)] \]
Ramond sector:
(T. Erler, S. Konopka, I.S. to appear)

\[ \Psi_R = c e^{-\Phi_2/4} \text{matter} = \Psi_R^{(-\frac{1}{2})} \]

Free theory is problematic: \( S_0(\Psi_R) = \omega^a(\Psi_R, Q \Psi_R) \equiv 0 \)!

try \( \omega(\Psi_R, Y Q \Psi_R) \), but \( x \) is degenerate off-shell \( \Rightarrow Y \) ill defined

1) impose constraint on \( H_0 \) s.t. \( Y \) exists
   but, what about \( \text{coh}(Q) = \ker(Q)/Q(\cdot) \) reduced ?

2) can choose a gauge s.t. zero modes of \( x, y \) are absent
   but what about B-V ?

We will circumvent this problem by focussing on the e.m.

\( \rightarrow \) no quantization but

- tree-level amplitude
- space-time SUSY of class. sol's
- e.m. with R-R fields
**Ansatz**: 

\[ O = Q \phi_N + M_2(\phi_N, \phi_N) + m_2(\psi_R, \psi_R) + \cdots \]

\[ O = Q \psi_R + M_2(\phi_N, \psi_R) + M_2(\psi_R, \phi_N) + \cdots \]

more generally, if there are less than 2 Ramond inputs, we take \( M_n \) from the \( NS \)-sector. For 2 or more Ramond inputs we need new products:

e.g. \( [Q, M_3](R,R,R) + M_2 \cdot m_2(R,R,R) \stackrel{B-V}{\Rightarrow} 0 \)

sol'n: \( \tilde{M}_3(R,R,R) = -\mu_2 \cdot m_2(R,R,R) \)

then, \( [Q, \tilde{M}_4](R,R,R,R) + [\tilde{M}_3, m_2](R,R,R,R) = 0 \)

Thus, \( \tilde{M}_4 \equiv 0 \), no \( (R,R,R,R) \)-interaction in SFT!

The formulas already determine the general structure:
\[ M_n \big|_0 = M_n^{NS} \circ P_0 \quad \text{projects on } \emptyset \text{ or } 1 \text{ Ramond input} \]
\[ M_n \big|_0 = M_n^{NS} \circ P_0 \quad \text{projects on } 2 \text{ or } 3 \text{ R's} \]
\[ \tilde{M}_n \big|_2 = \tilde{M}_n \circ P_z \quad \text{new} \]

Recursion relation:
\[
\tilde{M}_{n+3} \big|_2 = \frac{1}{n+1} \sum_{k=0}^{n} \left[ \tilde{M}_{k+2} \big|_2 , M_{n-k+2} \big|_0 \right]
\]
equivalently:

\[ M(t) = \sum_{n=0}^{\infty} t^n M_{n+1} \bigg|_0 \]

\[ m(t) = \sum_{n=0}^{\infty} t^n m_{n+2} \bigg|_0 \]

\[ \mu(t) = \sum_{n=0}^{\infty} t^n \mu_{n+2} \bigg|_0 \]

\[ \widetilde{M}(t) = \sum_{n=0}^{\infty} t^n \widetilde{M}_{n+2} \bigg|_2 \]

then:

\[ \frac{d}{dt} M(t) = [M(t), \mu(t)] \quad ; \quad \frac{d}{dt} \widetilde{M}(t) = [\widetilde{M}(t), \mu(t)] \]

\[ \frac{d}{dt} m(t) = [m(t), \mu(t)] \quad ; \quad \mu(t) = \xi \cdot m(t) \]
Summary:

1) The NS sector of open superstring field theory is a "large" closed string \( L_\infty \) gauge transformation of the free theory, \( S = \omega(\Phi,Q\Phi) \).

2) NS+R OSSFT is given by the same (modulo projectors) of \( Q + m_2 \), \( m_2 = 4 \) (Witten).

\[ \therefore \] The NS+R vertices are not cyclic (i.e. do not come from an action). But the S-matrix derived from them is cyclic (S. Konopka, to appear).

3) This establishes the existence of a decomposition of super moduli space at the level of CFT.

4) Type II and heterotic string works out similarly (Erler, Konopka, I.S., '14)