Scattering Inequalities

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Scattering Amplitudes

- Basic objects in Quantum Field Theory (QFT)
- Predictions for colliders: cross-sections
- My motivation: new ideas in QFT
Perturbative QFT

- Loop expansion

- **Integrand**: rational function before integration

\[ I(\ell_j, k_i, s_i) \text{ sum of Feynman diagrams} \]

\[ \Omega = d^4 \ell_1 \ldots d^4 \ell_L I(\ell_j, k_i, s_i) \quad A = \int_{\ell_j \in \mathbb{R}} \Omega \]

Integrand form
Integrand

- Ideal object to study: finite, well-defined
- Fixed by principles of QFT
- Qualitative information about the final amplitudes
  - Collinear limits: IR divergencies
  - Poles at infinity: UV structure
  - Types of singularities: transcendental properties
Feynman diagrams

- Gauge redundancy: off-shell virtual particles
- Two principles manifest:

I) Locality: particles interact point-like

Amplitude: only poles
\[ \frac{1}{P^2} \rightarrow \infty \]

II) Unitarity: sum of probabilities is 1

Amplitude: factorization
\[ P = \sum_{i \in \sigma} p_i \]
Modern methods

- Re-express the integrand in the basis of integrals
- Fix coefficients using cuts: \( I = \sum_j c_j I_j \)
- Unitarity cuts:
  \[ \ell^2 = (\ell + Q)^2 = 0 \]
- Maximal cuts, leading singularities:
Planar limit

- The integrand defined as a sum of diagrams
  - No global loop momenta
  - Each diagram: its own labels

- Planar limit: dual variables
  \[ k_1 = (x_1 - x_2) \quad k_2 = (x_2 - x_3) \]
  \[ \ell_1 = (x_3 - y_1) \quad \ell_2 = (y_2 - x_3) \]

Global labels
Integrand: single function
## Conditions on the amplitude

<table>
<thead>
<tr>
<th>Standard methods</th>
<th>Alternative</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Planar diagrams</td>
<td>• Same set of conditions</td>
</tr>
<tr>
<td>• Match physical cuts/singularities</td>
<td>• Packaged in a different way</td>
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- **Locality + Planarity**
- **Unitarity**

\[
\text{Cut}(I) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}
\]

Construction not known in general

Complete set known
Maximally supersymmetric Yang-Mills theory in planar limit

- “Simplest Quantum Field Theory”
- Conformal + dual conformal, convergent series
- Toy model for QCD
  - Tree-level amplitudes identical
  - Loop amplitudes simpler, no confinement
- Past: new methods for amplitudes originated here
Many faces of the theory

- Useful playground for many theoretical ideas

Diagram:
- Integrability
- Yangian
- AdS/CFT
- Strong coupling
- Twistor methods
- Hexagon bootstrap
- Amplitudes/Wilson loops
- OPE expansion
Integrand in planar $N=4$ SYM

- Superamplitudes $\mathcal{I}_{n,\ell}$

\[ \mathcal{I}_{n,\ell} = \sum_{k} \tilde{\eta}^{4k} I_{n,k,\ell} \]

- Dual conformal symmetry

  (Drummond, Henn, Korchemsky, Sokatchev 2006)

  - Integral basis: no triangle subdiagrams
    = no poles at infinity momentum

- Recursion relations using on-shell diagrams

  (Arkani-Hamed, Bourjaily, Cachazo, Goncharov, Postnikov, JT 2012)

  - Logarithmic singularities

\[ \Omega \sim \frac{dx}{x} \quad \text{near} \quad x = 0 \]
The Amplituhedron

(Arkani-Hamed, JT 2013)
Volume of polyhedron

(Hodges 2009)

- New kinematical variables — momentum twistors $Z \in \mathbb{C}^3$
- Tree-level process: $gg \rightarrow 5g$
- Comparison of two calculations of recursion relations
Evidence for a new structure

**Volume of polyhedron**


gg → gg ... g

at tree-level

\[
\int_{\tilde{P}_n} \frac{D^4 W}{(Z_0 \cdot W)^5}
\]

Amplitude = volume

(Arkani-Hamed, Bourjaily, Cachazo, Hodges, JT 2010)

**Grassmannian**

Configurations of k-planes in n dimensions

All-loop order information

(Arkani-Hamed, Cachazo, Cheung, Kaplan 2009)
“Conjecture”

Amplitudes are volumes of some regions in some space
Strategy

- Simple intuitive geometric ideas: use equations
- Generalization:  
  - More complicated geometry
  - Higher dimensions
- Same equations persist
Road to Amplituhedron

Start:
Point inside a convex polygon
Road to Amplituhedron

**Start:**
Point inside a convex polygon
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**Start:** Point inside a convex polygon

Amplituhedron $\mathcal{A}_{n,k,\ell}$

A $k$-dim plane and $\ell$ lines inside a $(k+4)$-dim convex space defined by $n$ vertices
The Amplituhedron

- Volume of $A_{n,k,\ell}$:

  Amplitudes in maximally supersymmetric Yang-Mills theory

  $\ell = 0$: Amplitudes of gluons in QCD

- Consistency check: Locality and Unitarity

- Explicit checks against reference theoretical data
Volume of the space

- Differential form with logarithmic singularities

- Simple examples:

  \[ x > 0 : \quad \text{Vol} = \frac{dx}{x} \]

  \[ y > 0, x > 0 : \quad \text{Vol} = \frac{dx}{x} \frac{dy}{y} \]

  \[ y > x > 0 : \quad \text{Vol} = \frac{dx}{x} \frac{dy}{y - x} \]
The Amplituhedron

- In the definition of Amplituhedron

\[ \mathcal{Y} = C \cdot Z \]

Amplituhedron \quad Positive matrices: \quad \begin{vmatrix} * & * \end{vmatrix} > 0

Minors are positive

- Positivity: crucial property of geometry
  - Locality, unitarity, even planarity derived from it
  - Hidden symmetry of this theory (Yangian) manifest
Inequalities

- Amplituhedron variables $z_i$

  $$(p_i, \epsilon_j, \ell_k) \rightarrow (x_i, \tilde{\eta}_j, y_k) \rightarrow (Z_i, \eta_j, Z^{(k)}_{AB}) \rightarrow z_i$$

- The definition of Amplituhedron: inequalities

  $$P_j(z_i) \geq 0$$

- Boundaries of the space

  $$P_j(z_i) = 0$$
Legal and illegal boundaries

• Singularities and cuts of the amplitude: localize $z_i$

• Inequalities hold $P_j(z_i) \geq 0$  \hspace{1cm} \ell_k \in \mathbb{C} \leftrightarrow z_i > 0$
  
  - Point inside the Amplituhedron space
  - Physical cut or singularity of the amplitude

• One or more inequalities violated $P_j(z_i) < 0$
  
  - Point outside the Amplituhedron space
  - Unphysical cut or singularity of the amplitude
Example 1: One-loop amplitude

- Consider 4pt one-loop amplitude
- Inequalities: \( z_1, z_2, z_3, z_4 \geq 0 \)
- Boundaries of the space: \( z_1, z_2, z_3, z_4 = (0, \infty) \)
- Differential form

\[
\Omega = \frac{d\,z_1}{z_1} \frac{d\,z_2}{z_2} \frac{d\,z_3}{z_3} \frac{d\,z_4}{z_4}
\]
Example 1: One-loop amplitude

- Cuts of the amplitude

![Diagram of a one-loop amplitude]

1 2 3 4
Example 1: One-loop amplitude

- Cuts of the amplitude

\[ z_1 = 0 \]
Example 1: One-loop amplitude

- Cuts of the amplitude

\[
\begin{align*}
z_1 &= 0 \\
z_2 &= 0
\end{align*}
\]
Example 1: One-loop amplitude

- Cuts of the amplitude

\[ z_1 = 0 \]
\[ z_2 = 0 \]
\[ z_3 = 0 \]
Example 1: One-loop amplitude

- Cuts of the amplitude

\[ z_4 = 0 \]

\[ z_1 = 0 \]
\[ z_2 = 0 \]
\[ z_3 = 0 \]
Example 1: One-loop amplitude

- Cuts of the amplitude

- $z_4 = 0$

- $z_1 = 0$
  $z_2 = 0$
  $z_3 = 0$

- $z_4 = \infty$

- $\ell \to \infty$

- $z_4 \in \mathbb{C}$

  "no-triangle"
Example 2: Two-loop amplitude

- Consider 4pt two-loop amplitude

- Inequalities:
  \[ z_1, z_2, z_3, z_4 \geq 0 \]
  \[ z_5, z_6, z_7, z_8 \geq 0 \]
  \[ (z_1 - z_5)(z_6 - z_2) + (z_3 - z_7)(z_8 - z_4) \geq 0 \]
Example 2: Two-loop amplitude

- Consider 4pt two-loop amplitude

- Inequalities:  
  \[ z_1, z_2, z_3, z_4 \geq 0 \]
  \[ z_5, z_6, z_7, z_8 \geq 0 \]
  \[ (z_1 - z_5)(z_6 - z_2) + (z_3 - z_7)(z_8 - z_4) \geq 0 \]

- Check: one-loop cut
  \[ z_1 = 0 \]
  \[ z_2 = 0 \]
  \[ z_3 = 0 \]
  \[ z_4 = 0 \]

\[ -z_5 z_6 - z_7 z_8 \geq 0 \]

Ω vanishes on this cut
Example 3: Unitarity cut

- Standard formulation

\[
P_j(z_i) \geq 0 \quad z_1 = z_2 = 0
\]

\[
P_j^{(1)}(z_i) \geq 0 \quad i = 3, \ldots, k
\]

\[
P_j^{(2)}(z_i) \geq 0 \quad i = k + 1, \ldots, m
\]

where \(k\) is a free parameter

- Set of inequalities split into two sets

\[
\text{Cut } M_{n,\ell} = \sum_{\ell_1 + \ell_2 = \ell - 1} M_{n_1,\ell_1} M_{n_2,\ell_2}
\]
Physics vs geometry

**Standard methods**
- Planar diagrams
- Locality + Planarity
- Match physical cuts/singularities

**Amplituhedron**
- Inequalities
  \[ P_j(z_i) \geq 0 \]
- Logarithmic form
  \[ \Omega \sim \frac{dx}{x} \]

Construction not known in general

Complete set known
Matching zeroes

(Arkani-Hamed, Hodges, JT 2014)

- Inequalities $P_j(z_i) \geq 0 \rightarrow$ Unique form $\Omega$

- Smaller set of information fixes the form

\[ \Omega = \frac{N(z_i)}{D(z_i)} \]

up to an overall constant

- Points outside the Amplituhedron
- Points inside with multiple poles

- Checked explicitly for several examples
Summary of the planar part

• Amplituhedron \( P_j(z_i) \geq 0 \)
  - Points on boundaries — physical poles
  - Points outside — unphysical poles

• Differential form \( \Omega \)
  
  **Standard:**
  - Physical, logarithmic poles \( \sim \frac{dx}{x} \)
  - No poles outside Amplituhedron
  - Residues on the inside: reduced inequalities

  **Conjecture:**
  - Yes
  - Yes
  - Redundant

Homogeneous problem
Generalizations

Next step:

- Stay planar, go beyond $\mathcal{N} = 4$ SYM
- Go to non-planar $\mathcal{N} = 4$ SYM
Generalizations

Next step:

- Stay planar, go beyond $\mathcal{N} = 4$ SYM
- Go to non-planar $\mathcal{N} = 4$ SYM

In this talk
Non-planar amplitudes

(Arkani-Hamed, Bourjaily, Cachazo, JT 2014)
(Bern, Herrmann, Litsey, Stankowicz, JT 2014 + in progress)
No global variables

- Absence of global variables
- We cannot guess/test inequalities immediately
- Check implications for the amplitude form $\Omega$

What is $\ell$?
Non-planar form

- Use of standard momenta \( k_i, \ell_k \)
- No single form, sum of diagrams
  \[
  \Omega = \sum_{\sigma, j} C_j \cdot \Omega_j(k_i, \ell_k)
  \]
  \( C_j \) color factor
- Each has its own variables
Constraints

- Inspired by the planar sector we conjecture:
  - Logarithmic singularities \( \Omega \sim \frac{dx}{x} \)
  - No poles at \( \ell \rightarrow \infty \)

- Stronger condition: each diagram individually
  \[
  I_j(k_i, \ell_k) = \frac{N_j(k_i, \ell_k)}{P^2_1 P^2_2 \ldots P^2_m}
  \]

- Find the basis and expand the amplitude
Evidence 1: Two-loop amplitude

Expansion of the 4pt two-loop amplitude

Two basis integrals

\[ N_1 = (k_1 + k_2)^2 \]
\[ N_2 = (k_1 + k_2)^2 \]

Double Poles

Poles at infinity
Evidence 1: Two-loop amplitude

Expansion of the 4pt two-loop amplitude

Two basis integrals

\[ N_1 = (k_1 + k_2)^2 \]

\[ N_2 = (k_1 + k_2)^2 \]

Double Poles

\( \text{NO} \) \hspace{1cm} \text{YES} \)

Poles at infinity

\( \text{NO} \) \hspace{1cm} \text{YES} \)

(Bern, Rozowsky, Yan 1997)
Evidence 1: Two-loop amplitude

\[
dI = \frac{d^4 \ell_1 \, d^4 \ell_2 \, (p_1 + p_2)^2}{\ell_1^2 (\ell_1 - k_2)^2 (\ell_1 - k_1 - k_2)^2 \ell_2^2 (\ell_2 - k_3)^2 (\ell_1 + \ell_2)^2 (\ell_1 + \ell_2 + k_4)^2}
\]

Perform cuts \( \ell_2^2 = (\ell_2 - k_3)^2 = (\ell_1 + \ell_2)^2 = (\ell_1 + \ell_2 + k_4)^2 = 0 \)

Localize \( \ell_2 \) completely
**Evidence 1: Two-loop amplitude**

\[
\text{Cut}_1 \, dI = \frac{d^4 \ell_1}{\ell_1^2 (\ell_1 - k_2)^2 (\ell_1 - k_1 - k_2)^2 [ (\ell_1 + k_3)^2 (\ell_1 + k_4)^2 - \ell_1^2 (\ell_1 + k_3 + k_4)^2 ]}
\]

Localize \( \ell_1 = \alpha k_2 \) by cutting \( \ell_1^2 = (\ell_1 - k_2)^2 = 0 \) and the Jacobian
Evidence 1: Two-loop amplitude

\[ \text{Cut}_{1,2} \, dI = \frac{d\alpha}{(\alpha + 1)\alpha^2tu} \]

- Double pole for \( \alpha = 0 \)
- There is also pole at infinity
- We want to find a numerator which cancels all that
Evidence 1: Two-loop amplitude

\[ \text{Double pole for } \alpha = 0 \]

New numerator

\[ N = (\ell_1 + k_3)^2 + (\ell_1 + k_4)^2 \]

Cancels double pole

\[ N \rightarrow \alpha s \]
Evidence 1: Two-loop amplitude

- New expansion of the 4pt two-loop amplitude
  (Arkani-Hamed, Bourjaily, Cachazo, JT, 2014)

Two basis integrals

\[ N_1 = (k_1 + k_2)^2 \]
\[ N_2 = (\ell_1 + k_3)^2 + (\ell_1 + k_4)^2 \]

Double Poles: NO  NO
Poles at infinity: NO  NO

Expand amplitude in the basis: YES
Evidence 2: Three-loop amplitude

Basis for three-loop four point amplitude

(Bern, Carrasco, Dixon, Johansson, Kosower 2007)

<table>
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<tr>
<th>Numerator</th>
<th>Double pole</th>
<th>Pole at infinity</th>
</tr>
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<tbody>
<tr>
<td>Original</td>
<td>YES</td>
<td>YES</td>
</tr>
<tr>
<td>BCJ</td>
<td>YES</td>
<td>YES</td>
</tr>
</tbody>
</table>
Evidence 2: Three-loop amplitude

- Basis for three-loop four point amplitude

Old numerator

\[ N = (\ell_5 + k_4)^2(k_1 + k_2)^2 \]
Evidence 2: Three-loop amplitude

- Basis for three-loop four point amplitude

\[ N = (\ell_5 + k_4)^2(k_1 + k_2)^2 \]

Old numerator

\[ N = (\ell_5 + k_4)^2[(\ell_5 + k_3)^2 + (\ell_5 + k_4)^2] \]

New numerator
Evidence 2: Three-loop amplitude

Basis for three-loop four point amplitude

(Bern, Herrmann, Litsey, Stankowicz, JT 2014)

Numerator | Double pole | Pole at infinity
---|---|---
Original | YES | YES
BCJ | YES | YES
New | NO | NO

Expansion of the amplitude: YES
Fixing coefficients

• **Standard approach:**
  - Unitarity cut
  - Maximal cut
  - Leading singularity
  - Non-zero RHS
  - \( \text{Cut}(I) = \ldots \)

• **Proposal:**

  Illegal cuts \( \text{Cut}(I) = 0 \) fix uniquely result!
  (up to an overall constant)
Explicit check

- Two-loop amplitude

\[ M_2 = \sum_{\sigma} a_1 + a_2 \]

Illegal 5-cut

Fixes relative coefficient

\[ a_1 = a_2 \]

Also three-loop construction

\[ k = 1 \]
Non-planar summary

- Absence of global variables: no inequalities yet
- Test of implications:
  - Logarithmic form
  - No poles at infinity
  - Diagrams + Zeroes fix the answer

Homogeneous conditions
Outlook

- **Final conjecture:**
  - Logarithmic singularities
  - Fixed by zeroes

  Amplitudes in $\mathcal{N} = 4$ SYM are fixed by homogeneous conditions

- **Future directions:**
  - Search for global variables
  - Inequalities and geometric interpretation
  - Exploring $\mathcal{N} = 8$ SUGRA and $\mathcal{N} < 4$ SYM
Thank you for your attention