Chern-Simons Theory, S-duality, and a Tridiagonal Determinant Identity

An equivalence between two Hilbert spaces will be developed: (i) the space of states of $U(1)^n$ Chern-Simons theory on $T^2$ with coupling constants given by a certain class of tridiagonal matrices (with corners); and (ii) the space of ground states of strings on an associated mapping torus with $T^2$ fiber. The equality of dimensions of the two Hilbert spaces (i) and (ii) is equivalent to a known identity on determinants of tridiagonal matrices with corners. The equivalence of operator algebras acting on the two Hilbert spaces follows from a relation between the Smith normal form of the Chern-Simons coupling constant matrix and the isometry group of the mapping torus, as well as the torsion part of its first homology group. The equivalence follows by studying the space of ground states of $SL(2,\mathbb{Z})$-twisted circle compactifications of $U(1)$ gauge theory, connected with a Janus configuration, and further compactified on a torus. Extension to $U(n)$ gauge theory will also be discussed.