Rigid supersymmetric backgrounds of Newton-Cartan supergravity

James Liu

University of Michigan

13 April 2016

Knodel, Lisbão and JTL, arXiv:1512.04961

Outline

- Introduction/Motivation
  - Coupling field theories to gravity
- Newton-Cartan gravity and supergravity
  - Three-dimensional $\mathcal{N} = 2$ supergravity
- Rigid supersymmetric backgrounds
  - Maximally supersymmetric and 1/2 BPS
- Concluding remarks

Relativistic field theories in curved backgrounds

- There is a natural procedure to couple relativistic field theory to a curved background

$$\eta_{\mu\nu} \to g_{\mu\nu}, \qquad \partial_\mu \to \nabla_\mu$$

- Consider, for example

$$S = \int d^4x[-\tfrac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}m^2\phi^2]$$

$$\Rightarrow \quad S = \int \sqrt{-g}\, d^4x[-\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}m^2\phi^2]$$

- Why is this useful?
  - Needed for theories with dynamical gravity
  - Natural means to investigate the Weyl anomaly under $g_{\mu\nu} \to e^{2\sigma}g_{\mu\nu}$

Fixing the curvature coupiings

- But not all couplings are uniquely determined

$$S = \int \sqrt{-g}\, d^4x [-\tfrac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \tfrac{1}{2} m^2 \phi^2 - \tfrac{1}{6} R \phi^2]$$

- In some cases, the couplings are fixed by additional symmetries
  - minimal versus conformal coupling
  - comparison to string amplitudes (for effective string actions)
  - invariance under supersymmetry
- Many powerful results for supersymmetric field theories in a curved background
  - Supersymmetric partition functions (S^4) and Wilson loops
  - The superconformal index ($S^3 \times S^1$)
  - Many supersymmetric observables can be computed using localization

Rigid supersymmetric backgrounds

- How do we put a supersymmetric field theory in a curved background?
- Curved backgrounds are natural in supergravity theories
  - But we do not want dynamical gravity

- Take supergravity coupled to matter, and then freeze out gravity by taking $M_{pl} \to \infty$
  - Decouple the gravitino, but demand $\delta\psi_\mu = 0$
- This trick works best in an off-shell formalism with auxiliary fields
  - Gravitino variation does not include matter fields
  - There is no need to specify the supergravity equations of motion

[Festuccia and Seiberg, arXiv:1105.0689]

Non-relativistic field theories and AdS/CMT

- Can we couple non-relativistic theories to a curved background?
- Recent interest motivated by non-relativistic holography

$$\text{Lifshitz:} \qquad ds^2 = -\frac{dt^2}{r^{2z}} + \frac{d\vec{x}^2 + dr^2}{r^2}$$

$$\text{Schrödinger:} \qquad ds^2 = -\frac{dt^2}{r^{2z}} + \frac{2dt\, d\xi + d\vec{x}^2 + dr^2}{r^2}$$

- We can perform a Wigner-Inönü contraction to obtain non-relativistic symmetry

Poincaré		$\longrightarrow$	Galilei	
translations	P_μ		time translation	H
			space translation	P_i
Lorentz	$M_{\mu\nu}$		rotations	J_{ij}
			Galilean boosts	G_i

Coupling a Galilean theory to gravity

- Consider the free action [Son, arXiv:1306.0638]

$$S = \int dt\, d^d x \left[\frac{i}{2} \Phi^\dagger \overleftrightarrow{D}_0 \Phi + \frac{\delta^{ij}}{2m} D_i \Phi^\dagger D_j \Phi \right]$$

Introduce a space metric $h^{\mu\nu}$ and a velocity vector v^μ

$$S = \int d^{d+1}x \sqrt{\gamma} \left[\frac{i}{2} v^\mu \Phi^\dagger \overleftrightarrow{D}_\mu \Phi + \frac{h^{\mu\nu}}{2m} D_\mu \Phi^\dagger D_\nu \Phi \right]$$

$$\gamma^{\mu\nu} = -v^\mu v^\nu + h^{\mu\nu}$$

- This can be made more systematic

  $\Rightarrow$ Newton-Cartan geometry
  - A reformulation of Newtonian gravity in a coordinate invariant manner
  - Can be obtained as a non-relativistic limit of general relativity

Newton-Cartan gravity

- Take the $c \to \infty$ limit of Einstein gravity

$$g_{\mu\nu} = \begin{pmatrix} c^2 g_{00} & c\, g_{0i} \\ c\, g_{i0} & g_{ij} \end{pmatrix} \quad \to c^2 \begin{pmatrix} g_{00} & 0 \\ 0 & 0 \end{pmatrix} = c^2 \tau_{\mu\nu}$$

$$g^{\mu\nu} = \begin{pmatrix} g^{00}/c^2 & g^{0i}/c \\ g^{i0}/c & g^{ij} \end{pmatrix} \to \quad \begin{pmatrix} 0 & 0 \\ 0 & g^{ij} \end{pmatrix} = h^{\mu\nu}$$

- Writing the temporal metric as $\tau_{\mu\nu} = \tau_\mu \tau_\nu$ with $h^{\mu\nu}\tau_\nu = 0$ gives the Galilei structure $\boxed{(\mathcal{M}, \tau_\mu, h^{\mu\nu})}$

τ_μ : specifies time direction

$h^{\mu\nu}$: (degenerate) spatial metric

- Introducing a vector v^μ satisfying $\tau_\mu v^\mu = 1$ gives the Newton-Cartan structure $\boxed{(\mathcal{M}, \tau_\mu, v^\mu, h^{\mu\nu})}$

v^μ : velocity vector

The central extension

- This now provides enough data to define a unique 'lower' metric $h_{\mu\nu}$ satisfying

$$h_{\mu\nu} h^{\nu\rho} = \delta_\mu^\rho - \tau_\mu v^\rho \qquad h_{\mu\nu} v^\nu = 0$$

(Note that $h_{\mu\nu}$ and $h^{\mu\nu}$ are not matrix inverses of each other, as they are not invertible)

- It is useful to extend the Galilei algebra to the Bargmann algebra by introducing

$$[G_i, P_j] = -\delta_{ij} M$$

where the central generator M is related to the mass

- Gauging M then introduces a $U(1)$ current, and extends the Newton-Cartan structure to $\boxed{(\mathcal{M}, \tau_\mu, v^\mu, h^{\mu\nu}, m_\mu)}$

Torsional Newton-Cartan geometry

- Motivated by holography we classify the Newton-Cartan geometries according to

$d\tau = 0$	Newton-Cartan (NC)
$d\tau \neq 0$ but $\tau \wedge d\tau = 0$	twistless Newton-Cartan (TTNC)
$\tau \wedge d\tau \neq 0$	torsional Newton-Cartan (TNC)

  [Christensen, Hartong, Obers and Rollier, arXiv:1311.4794]

- Here we focus only on the torsionless $d\tau = 0$ case
  - We can introduce adapted coordinates by defining the time coordinate by $\tau = dt$

$$\tau_\mu = (1, 0), \qquad v^\mu = (1, v^i)$$

$$h^{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & h^{ij} \end{pmatrix}, \qquad h_{\mu\nu} = \begin{pmatrix} h_{kl} v^k v^l & -h_{jk} v^k \\ -h_{ik} v^k & h_{ij} \end{pmatrix}$$

where $h_{ij} = (h^{ij})^{-1}$

Working with adapted coordinates

- In the torsionless case, we can introduce a metric compatible connection
$$\nabla_\mu h^{\nu\lambda} = 0 \qquad \nabla_\mu \tau_\nu = 0$$
where
$$\Gamma^\rho{}_{\mu\nu} = v^\rho \partial_{(\mu}\tau_{\nu)} + \tfrac{1}{2} h^{\rho\lambda}(\partial_\nu h_{\lambda\mu} + \partial_\mu h_{\lambda\nu} - \partial_\lambda h_{\mu\nu} + 2K_{\lambda(\mu}\tau_{\nu)})$$
$$K_{\mu\nu} = 2\partial_{[\mu} m_{\nu]}$$

- In the adapted coordinates, we find $\Gamma^0{}_{\mu\nu} = 0$ along with
$$\begin{aligned}\Phi^i \equiv \Gamma^i{}_{00} &= h^{ij}(\partial_i\varphi - \partial_0 A_i)\\ \Omega^i{}_j \equiv \Gamma^i{}_{0j} &= \tfrac{1}{2} h^{ik}(\partial_0 h_{kj} + \partial_k A_j - \partial_j A_k)\\ \Gamma^i{}_{jk} &= \tfrac{1}{2} h^{il}(\partial_j h_{lk} + \partial_k h_{lj} - \partial_l h_{jk})\end{aligned}$$
where
$$\varphi = m_0 - \tfrac{1}{2} h_{ij} v^i v^j \qquad A_i = m_i + h_{ij} v^j$$

[Andringa, Bergshoeff, Panda and de Roo, arXiv:1011.1145]

Newtonian dynamics

- In adapted coordinates, the geodesic equation reduces to

$$\frac{d^2x^i}{dt^2} + \Phi^i + 2\Omega^i{}_j \frac{dx^j}{dt} + \Gamma^i{}_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0$$

- Newtonian gravity is obtained by taking the Einstein equation

$$R_{\mu\nu} = 4\pi G\rho\, \tau_\mu \tau_\nu$$

  - The $R_{ij} = 0$ component implies $\Gamma^i{}_{jk} = 0$
  - We can further make a coordinate transformation to set $h^{ij} = \delta^{ij}$ and $\Omega^i{}_j = 0$ so that

$$\boxed{\frac{d^2x^i}{dt^2} = -\partial_i\varphi \qquad \text{along with} \qquad R_{00} = \nabla^2\varphi = 4\pi G\rho}$$

where $\Phi^i = \partial_i\varphi$

Introduction of supersymmetry

- Can we supersymmetrize Newton-Cartan gravity?

  Gauge a super-Bargmann algebra

symmetry	generator	gauge field
time translation	H	τ_μ
space translation	P^a	e_μ^a
rotations	J^{ab}	${\omega_\mu}^{ab}$
Galilean boosts	G^a	${\omega_\mu}^{a}$
central charge	M	m_μ
supersymmetries (3D)	$Q_\pm$	$\psi_{\mu\pm}$

- The on-shell 3-dimensional $\mathcal{N} = 2$ theory was constructed in Andringa, Bergshoeff, Rosseel and Sezgin, arXiv:1305.6737

Obtaining 3-dimensional Newton-Cartan supergravity

- We can also start from the relativistic theory and take the non-relativistic limit
  - Take the off-shell $\mathcal{N} = (2, 0)$ theory [Howe, Izquierdo, Papadopoulos and Townsend, hep-th/9505032] as the starting point
  - Then perform a Wigner-Inönü contraction

vielbein	E_μ^A	$\rightarrow$	$\tau_\mu, \quad e_\mu^a$
gravitini	$\Psi_{\mu\, i}$	$\rightarrow$	$\psi_{\mu\pm}$
central charge gauge field	M_μ	$\rightarrow$	m_μ
R-charge gauge field	V_μ		(eliminated)
auxiliary scalar	D	$\rightarrow$	S

[Bergshoeff, Rosseel and Zojer, arXiv:1505.02095]

- This reproduces the on-shell result, but includes the auxiliary field S

The transformation rules

- The supersymmetry transformation rules

$$
\begin{aligned}
\delta\tau_\mu &= \tfrac{1}{2}\bar{\epsilon}_+\gamma^0\psi_{\mu+} \\
\delta e_\mu^a &= \tfrac{1}{2}(\bar{\epsilon}_+\gamma^a\psi_{\mu-} + \bar{\epsilon}_-\gamma^a\psi_{\mu+}) \\
\delta m_\mu &= \bar{\epsilon}_-\gamma^0\psi_{\mu-} \\
\delta\psi_{\mu+} &= (D_\mu + S\tau_\mu\gamma_0)\epsilon_+ \\
\delta\psi_{\mu-} &= (D_\mu - 3S\tau_\mu\gamma_0)\epsilon_- + (\tfrac{1}{2}\omega_\mu{}^a\gamma_{a0} - Se_\mu^a\gamma_a)\epsilon_+ \\
\delta S &= -\tfrac{1}{8}\bar{\epsilon}_+\gamma^{a0}\hat{\psi}_{a0-}
\end{aligned}
$$

- For a supersymmetric background, we set $\psi_{\mu+} = \psi_{\mu-} = 0$, and solve the Killing spinor equations $\delta\psi_{\mu+} = \delta\psi_{\mu-} = 0$
  - Goal: obtain a solution for $(\tau_\mu, e_\mu^a, m_\mu, S)$

Integrability of the Killing spinor equation

- The Killing spinor equations are given by

$$
\begin{aligned}
D_\mu \epsilon_+ &= -S\tau_\mu \gamma_0 \epsilon_+ \\
D_\mu \epsilon_- &= 3S\tau_\mu \gamma_0 \epsilon_- - \tfrac{1}{2}\omega_\mu^a \gamma_{a0}\epsilon_+ + S e_\mu^a \gamma_a \epsilon_+
\end{aligned}
$$

- The integrability condition $[D_\mu, D_\nu]\epsilon_\pm = 0$ then gives

$$
\begin{aligned}
A_{\mu\nu}\gamma_0\epsilon_+ &= 0 \\
B_{\mu\nu}\gamma_0\epsilon_- + C^a_{\mu\nu}\gamma_a\epsilon_+ &= 0
\end{aligned}
$$

where

$$
\begin{aligned}
A &= -\tfrac{1}{4}R^{ab}(J)\epsilon_{ab} - \tau \wedge dS \\
B &= -\tfrac{1}{4}R^{ab}(J)\epsilon_{ab} + 3\tau \wedge dS \\
C^a &= -\tfrac{1}{2}\epsilon^a{}_b R^b(G) + e^a \wedge dS - 2S^2 \epsilon^a{}_b e^b \wedge \tau
\end{aligned}
$$

Supersymmetric backgrounds

- To obtain a supersymmetric background, we solve

$$A\gamma_0\epsilon_+ = 0$$
$$B\gamma_0\epsilon_- + C^a\gamma_a\epsilon_+ = 0$$

for $\epsilon = (\epsilon_+, \epsilon_-)$ a four real component spinor

- We can consider the following cases
  1. Maximally supersymmetric:
     four independent Killing spinors $\Rightarrow A = B = C^a = 0$
  2. 1/2 BPS with $\epsilon = (0, \epsilon_-) \Rightarrow B = 0$
  3. 1/2 BPS with $\epsilon = (\epsilon_+, 0) \Rightarrow A = C^a = 0$
  4. 1/2 BPS with $\epsilon = (\epsilon_+, F\epsilon_+)$ for some function $F(x^\mu)$
     $\Rightarrow A = 0$ with B and C^a related

Maximal supersymmetry

- For maximal supersymmetry, we immediately obtain

$$\left.\begin{aligned} 0 = A = -\tfrac{1}{4}R^{ab}(J)\epsilon_{ab} - \tau \wedge dS \\ 0 = B = -\tfrac{1}{4}R^{ab}(J)\epsilon_{ab} + 3\tau \wedge dS \end{aligned}\right\} \quad \Rightarrow \quad R^{ab}(J) = 0, \quad \tau \wedge dS = 0$$

  as well as

$$C^a = 0 \quad \Rightarrow \quad R^a(G) = -2\epsilon^a{}_b e^b \wedge dS - 4S^2 e^a \wedge \tau$$

- The curvature in terms of rotation and boost spin connections can be combined to give Riemann in terms of the (non-relativistic) Christoffel connection

$$R^\mu{}_{\nu\rho\sigma}(\Gamma) = -8S^2\tau_\nu\tau_{[\rho}\delta^\mu_{\sigma]} + 4\epsilon^a{}_b e^\mu_a \tau_\nu e^b_{[\rho}\partial_{\sigma]}S$$

- Contracting gives $\qquad R_{\mu\nu} = 8S^2\tau_\mu\tau_\nu$
  - Newtonian dynamics with S^2 playing the role of mass density (but we have not imposed any equations of motion)

Maximal supersymmetry

- Supersymmetry imposes more than just the $R_{\mu\nu}$ equation
- Working in a frame with $h^{ij} = \delta^{ij}$ along with $\Gamma^i{}_{00} = \partial_i \varphi$ and $\Gamma^i{}_{0j} = 0$ gives

$$\partial_i \partial_j \varphi = 4S^2 \delta_{ij} \qquad \text{and} \qquad S = \text{const.}$$

- This is solved by taking the Newtonian potential

$$\varphi(t, x^i) = 2S^2 |x^i - \alpha^i(t)|^2 + \beta(t)$$

which is just a harmonic oscillator potential
  - 'Newtonian cosmology' with a uniform background mass density

Half-BPS with $\epsilon = (0, \epsilon_-)$

- For $\epsilon_+ = 0$ we have a single Killing spinor equaiton

$$(D_\mu - 3S\tau_\mu\gamma_0)\epsilon_- = 0 \qquad \Rightarrow \qquad R^{ab}(J) = 6\epsilon^{ab}\tau \wedge dS$$

- In adapted coordinates, this again implies $R_{ij} = 0$, so we may choose $h^{ij} = \delta^{ij}$
  - This time, however, we have $R_{0i}(\Gamma) = -6\epsilon_{ij}\partial_j S$, which allows for a larger family of solutions
- We can choose arbitrary $\Phi_i = \Gamma^i{}_{00}$ and $\Omega = \frac{1}{2}\epsilon_{ij}\Gamma^i{}_{0j}$ along with $S = \Omega/6 + f(t)$
- The Killing spinor is then

$$\epsilon_+ = 0 \qquad \epsilon_- = e^{3\gamma_0 \int f(t)dt}\epsilon_0$$

  - Note that this enhances to maximal supersymmetry when $S = \text{const.}$

Half-BPS with $\epsilon = (\epsilon_+, 0)$

- In this case, we have

$$(D_\mu + S\tau_\mu\gamma_0)\epsilon_+ = 0 \qquad \Rightarrow \qquad R^{ab}(J) = -2\epsilon^{ab}\tau \wedge dS$$
$$(\tfrac{1}{2}\omega_\mu^a\gamma_{a0} - Se_\mu^a\gamma_a)\epsilon_+ = 0$$

- The first equation again gives $R_{ij} = 0$ (so we choose $h^{ij} = \delta^{ij}$)
- Solving the algebraic equation then gives (in adapted coordinates)

$$S = -\tfrac{1}{2}\Omega + \tfrac{1}{4}\epsilon^{ab}\partial_a v_b$$
$$\partial_{(a}v_{b)} = 0$$
$$\partial_a m_0 - \partial_0 m_a = -\epsilon_{ab}v^b\epsilon^{cd}\partial_c m_d$$

- The Killing spinor is

$$\epsilon_+ = e^{-\frac{1}{4}\gamma_0\int\epsilon^{ab}\partial_a v_b\,dt}\epsilon_0, \qquad \epsilon_- = 0$$

The general half-BPS case

- For a solution of the form $(\epsilon_+, F\epsilon_+)$ we return to the Killing spinor equations

$$(D_\mu + S\tau_\mu\gamma_0)\epsilon_+ = 0 \qquad \Rightarrow \qquad R^{ab}(J) = -2\epsilon^{ab}\tau \wedge dS$$
$$(D_\mu - 3S\tau_\mu\gamma_0)\epsilon_- = -(\tfrac{1}{2}\omega_\mu^a\gamma_{a0} - Se_\mu^a\gamma_a)\epsilon_+$$

- As before, we solve the first equation by writing $h^{ij} = \delta^{ij}$ and taking $\partial_i(S + \frac{1}{2}\Omega) = 0$
- The Killing spinor ϵ_+ is given by

$$\epsilon_+ = e^{-\gamma_0 \int (S + \frac{1}{2}C)dt}\epsilon_0$$

while ϵ_- may be obtained by solving the inhomogeneous equation
  - This gives additional constraints on the background

What have we learned?

- In all cases, non-relativistic supersymmetry demands

$$R_{ij} = 0 \quad \Rightarrow \quad h^{ij} = \delta^{ij}$$

  - Is this special to $2+1$ dimensions, or does it also generalize to higher dimensions?
  - This is compatible with the Newtonian result $R_{\mu\nu} = 4\pi G \rho \tau_\mu \tau_\nu$, but we have not imposed any equations of motion
- This seems to forbid non-relativistic curved backgrounds such as $S^1 \times S^2$
  - In the relativistic case, such rigid supersymmetric backgrounds are obtained by turning on a vector auxiliary field in the time direction
  - Here we have m_μ, but it only enters indirectly through the connection

Next steps

- Couple matter multiplets to Newton-Cartan supergravity and take the rigid limit to obtain non-relativistic supersymmetric field theories
- Extend Newton-Cartan supergravity to higher dimensions [Bergshoeff et al]
- Framework for describing non-relativistic scale anomalies

$$\langle z T^t_t + T^i_i \rangle \neq 0$$

- Develop (torsional) Newton-Cartan geometry as a framework for Lifshitz or Schrödinger holography