Higher-Spin Symmetries in AdS/CFT Dualities and Phase Transitions

Dualities in Supergravities, Strings and Branes, Texas A&M

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Texas A&M, April, 13, 2016

Main Messages

Higher-Spin Symmetry is ∞-dim. extension of conformal (or AdS) symmetry and it is a crucial ingredient of (Vasiliev) HS theories and should be responsible for its consistency as a quantum theory

Higher-Spin theories do have CFT duals that describe some physics

By the AdS/CFT arguments Higher-Spin Symmetry should play a fundamental role on the CFT side as well, the fact that has remained obscure

Plan

- Unbroken HS symmetries in CFT: dramatic consequences for the correlation functions etc. Based on papers with N.Boulanger, S.Didenko, D.Ponomarev, M.Taronna.

- Higher-Spin Symmetries and Reality: Broken (more neutral version: Deformed) Higher-Spin Symmetries, anomalous dimensions in Wilson-Fisher CFT's. Based on [arXiv:1512.05994], see also Giombi and Kirillin.

HS Symmetry

It is difficult to say anything without defining HS symmetries. Several definitions will be given, but a simple analogy is

$$SU(N) \text{ is } SU(V) \text{ for finite } V$$

while HS algebra *hs*

$$hs \text{ is } SU(V) \text{ for certain infinite } V \sim \Box\phi = 0$$

at least in the simplest cases. Then it needs to be gauged, deformed, broken, ...

Symmetries of Free Boson

Let V be defined as the solution space of

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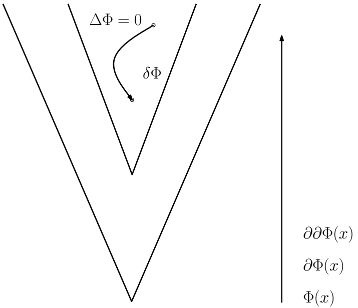
The maps $V \to V$ are symmetries of Klein-Gordon equation

$$\delta\Phi : \qquad \text{such that} \qquad \Box\delta\Phi\Big|_{\Box\Phi=0} = 0$$



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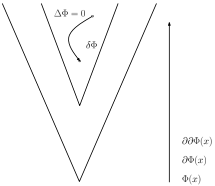
The equation is known to be conformally-invariant, so

$$\delta\Phi = v^m\partial_m\Phi + \frac{d-2}{2d}(\partial_m v^m)\Phi$$

where v^a is a Conformal Killing Vector

$$v^a = \epsilon^a + \epsilon^{a,b}x_b + \ldots$$



Higher Symmetries of Free Boson

Since the equation is a linear one

$$\Box\Phi = 0$$

a superposition of n symmetries is a symmetry again

$$\delta\Phi = \delta_{v_1}...\delta_{v_n}\Phi$$

These symmetries are differential operators with polynomial coefficients. For example, we find hyper-translations

$$\delta\Phi = \epsilon^{a_1...a_k}\partial_{a_1}...\partial_{a_k}\Phi$$

Not freely generated as there are trivial symmetries

$$\delta\Phi = \Box\Phi$$

HS Algebra-I

Following Eastwood we can define

$$hs = \text{all symmetries of } \Box\Phi = 0$$

The following facts are obviously true

- infinite-dimensional (e.g., all hyper-translations)
- associative (we can compose symmetries, there is no need in taking just the commutator)
- contains conformal algebra as a subalgebra

The last two points make one think of the universal enveloping algebra of the conformal algebra, $U(so(d,2))$.

Conformal symmetries are in one-to-one with $so(d,2)$ generators $T^{AB} = -T^{BA}$

$$\delta_v \Phi \Longleftrightarrow \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} = T^{AB} = so(d,2)$$

HS Algebra-II

Conformal symmetries are in one-to-one with $so(d,2)$ generators $T^{AB}=-T^{BA}$

$$\delta_v\Phi \Longleftrightarrow \ydiagram{1,1} = T^{AB} = so(d,2)$$

Superposition of symmetries = taking the tensor product

$$U\left(\ydiagram{1,1}\right) = \bullet \oplus \ydiagram{1,1} \oplus \left(\ydiagram{1,1,1,1} \oplus \ydiagram{2,2} \oplus \ydiagram{2} \oplus \bullet\right) \oplus \ydiagram{3,3} \oplus \ldots$$

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However, trivial symmetries lead to an ideal I (Eastwood)

$$hs = U\left(\ydiagram{1,1}\right)/I \qquad\qquad I = \ydiagram{1,1,1,1} \oplus \ydiagram{2} \oplus (\bullet - c)$$

HS Algebra I+II=III

There is an infinite family of conserved tensors $\partial^m J_{ma(s-1)} = 0$

$$J_s = \Phi\partial ... \partial\Phi + \text{properly symmetrized}$$

These tensors can be turned into Noether currents

$$j^m_v = J^m{}_{a(s-1)}\, v^{a(s-1)} \qquad v^{a(s-1)} - \text{Conformal Killing Tensor}$$

The currents are in one-to-one with the HS algebra generators

$$hs = \bullet \oplus \ydiagram{1,1} \oplus \ydiagram{2,2} \oplus \ydiagram{3,3} \oplus \ydiagram{4,4} \oplus ...$$

and can be used to built charges

$$Q_v = \int j_v$$

that form the same HS algebra

HS Algebra: 4d bonus

In lower dimensions there are exceptionally simple ~~theories of everything~~ realizations of HS algebras.

HS Algebra: 4d bonus

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$$\Box_3 \Phi = 0$$

is the same as the (even) operators on 2d Harmonic Oscillator

$$Y^{\mathcal{A}} = \{q^1, q^2, p_1, p_2\} \qquad\qquad [Y^{\mathcal{A}}, Y^{\mathcal{B}}] = C^{AB}$$

because $so(3,2) \sim sp(4)$ and $YY \sim sp(4)$

$$hs : F(Y) = F(-Y) = 1 + \overbrace{YY}^{10} + YYYY + \ldots$$

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For that reason one can often see HS people using Moyal star-product to multiply HS algebra elements:

$$(f \star g)(Y) = f(Y) \exp i(\overleftarrow{\partial}_{\mathcal{A}} C^{\mathcal{AB}} \overrightarrow{\partial}_{\mathcal{B}}) g(Y)\,,$$

Interlude

More generally, given any free CFT (e.g. in the form of a conformally invariant equation) there is an infinite-dimensional algebra behind it. It is generated by the infinite family of the HS conserved tensors, stress-tensor being the lowest member thereof. HS is a global symmetry. (Vasiliev; Boulanger, E.S.; Bekaert, Grigoriev;...)

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Can we revert the logic and look for CFT's that have HS symmetries? More symmetries the better!

Simplest implications of unbroken HS symmetry

$2d$ CFT's are governed by Virasoro, e.g. decoupling of $L_{-2}+\alpha L_{-1}^2$ imposed on $\langle O_\Delta O_{\Delta_1} O_{\Delta_2}\rangle$ relates Δ, Δ_1 and Δ_2

With HS symmetry $L_{-2}+\alpha L_{-1}^2$ gets replaced by ∂^ν, so the signature of HS symmetry is the presence of HS currents

$$\boxed{\partial^b J_{ba(s-1)} = 0}$$

Simplest three-point functions already tell something:

$$\langle J_s O_{\Delta_1} O_{\Delta_2}\rangle \qquad \Delta_1 = \Delta_2,\ s = 1, 2, \ldots$$

$$\langle J_s J_{s'} O_\Delta\rangle \qquad \Delta = 2\frac{d-2}{2}$$

The latter suggests $O = \Phi^2$ for free Φ. More nontrivial info is in OPE and Ward identities.

Not HS Ward identities

The conformal symmetry Ward identity reads:

$$-\int dS^{\underline{m}}\xi^a\langle J_{\underline{m}a}OO\rangle=\delta_\xi\langle OO\rangle$$

where ξ^a is a conformal Killing vector. In the differential form it is clear that the distribution is not well-defined and has to be regularized — otherwise currents are conserved for coincident points as well!

The stress-tensor Ward identity allows to recover the three-point functions from the two-point functions. Similar statement is true for the global $O(N)$ symmetry Ward identity.

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$$\langle J_s O_\Delta O_\Delta\rangle = g_{s00}(Q)^s\left(\frac{x_{23}^2}{x_{12}^2x_{13}^2}\right)^{\frac{d-2}{2}}\frac{1}{(x_{23}^2)^\Delta},$$

which leads to (Osborn, Petkou)

$$g_{001} = \frac{C_{OO}}{S_d}, \qquad\qquad g_{002} = \frac{C_{OO}\,d\Delta}{(d-1)S_d}.$$

HS Ward identities

HS current can be used to construct the actual current by contracting it with a Killing tensor (applies for $s = 2$ as well)

$$j_{\underline{m}}(v) = J_{\underline{m}a(s-1)} v^{a(s-1)} \qquad \partial^a v^{a(s-1)} - \text{traces} = 0$$

The simplest Killing tensor is a constant that induces

$$\delta\Phi = v^{a(s-1)} \partial_a ... \partial_a \Phi$$

The simplest Ward identity for these 'hyper-translations' is

$$-\int dS^{\underline{m}} \langle J_{\underline{m}a(s-1)} OO \rangle = \partial_a ... \partial_a \langle OO \rangle$$

and, as in $s = 1$ and $s = 2$ cases, fixes the coupling constant in

$$\langle J_{a(s)} OO \rangle \sim g_{s00} \times \text{standard structure}$$

CFT Invariants

The normalization of correlators is ambiguous, the only invariant for $00s$ is:

$$I_{s00} = \frac{\langle J_s O O\rangle^2}{\langle J_s J_s\rangle \langle O O\rangle^2}$$

For example, in $3d$ we have:

$$\begin{aligned} \text{boson}: &\qquad \frac{1}{N}\frac{2^{4-s}\Gamma\left(s+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma(s+1)} \\ \text{fermion}: &\qquad \frac{1}{s^2}\times \text{boson} \end{aligned}$$

The HS algebra is the same in $3d$ (exception!), but the representations are different, so are the invariants.

OPE—Representation— HS Algebra

The OPE of two HS currents is difficult to study

$$J_{s_1} J_{s_2} = O_2 + \sum_s J_s + \delta_{s_1,s_2} \langle JJ \rangle$$

If we integrate it once we get the action of charges on J

$$[Q_{s_1}, J_{s_2}] = \sum_s J_s$$

If we integrate one more time, we get the algebra

$$[Q_{s_1}, Q_{s_2}] = \sum_s Q_s$$

We assume stress-tensor J_2 and at least one HS current $J_{s>2}$.

Implications of unbroken HS symmetry

It can be proved via $[Q, J] = J$ or the Jacobi identity $[Q, [Q, Q]] = 0$, (Fradkin, Vasiliev; Anselmi; Maldacena, Zhiboedov; Boulanger, Ponomarev, E.S, Taronna; Alba, Diab; Stanev), that

(i) the presence of at least one HS charge requires all even at least, $s = 2, 4, 6, ...$;
(ii) the algebra is unique* in $d > 2$ except for $4d$;
(iii) in $4d$ there is a one-parameter family (Fradkin, Linetsky; Gunaydin; Boulanger, E.S.; Mkrtchyan^2, Manvelyan, Theisen);
(iv) the algebra can always be associated with a free field (not necessary scalar, e.g. $4d$ Maxwell)

$$\boxed{\text{unbroken HS} = \text{free CFT}}$$

The algebra of HS charges is what is gauged in HS theory (global symmetry of CFT = gauge symmetry in AdS).

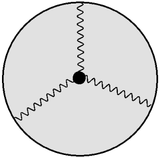
Higher-Spin Invariants

Since the HS algebras are so rigid and infinite-dimensional, it should be possible to recover all the correlation functions for a CFT having some HS symmetry just from the symmetry itself.

Sundell and Colombo suggested that

$$\langle J ... J\rangle = tr(\Phi \star ... \star \tilde{\Phi})$$

where Φ is the boundary-to-bulk propagator for the HS multiplet. In lower dimensions it turns out to be Gaussian (Giombi, Yin; Didenko, E.S.) and one can compute all



$$\langle JJJ\rangle = \frac{1}{|x_{12}||x_{23}||x_{31}|} \cos(Q^2_{13} + Q^3_{21} + Q^1_{32}) \times$$
$$\times \cos(P_{12}) \cos(P_{23}) \cos(P_{31})$$

Vasiliev-Vasiliev duality

The spectrum of the minimal bosonic HS theory is:

$$\Phi_0, \quad \Phi_{\underline{mm}}, \quad \Phi_{\underline{m}(4)}, \quad \ldots$$

The AdS/CFT relates it to the CFT correlators

$$\left\langle \exp \sum_s \int J_{a(s)} \phi^{a(s)} \right\rangle = \int D\Phi \Big|_{\Phi \to z^{3-\Delta}\phi} e^{-S[\Phi]}$$

For one choice of b.c. $4d$ Vasiliev HS theory is supposed to be dual to free CFT's (Sundborg; Konstein, Vasiliev, Zaikin; Sezgin-Sundell).

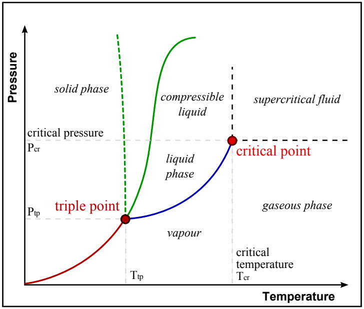
For another b.c. Klebanov and Polyakov, see also Sezgin and Sundell, at large-N related the same HS theory to the Wilson-Fisher CFT's, which were studied at large-N by Vasiliev et al; Lang, Ruhl; Petkou.

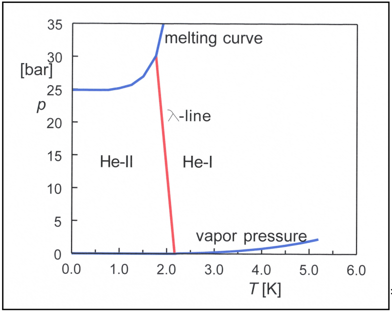
From exact to broken

Free CFT's are still nontrivial and should be dual to HS theories with b.c. preserving full HS symmetry. Still useful for AdS/CFT as an example of QG without extra dims and SUSY, but without post-Newtonian limit.

The HS symmetry is broken in Wilson-Fischer CFT's. The HS currents have small anomalous dimensions. It was argued (Giombi, Yin; Hartman, Rastelli) that the duality to critical model follows from the free case order by order in $1/N$.

Thanks to the renessaince of the bootstrap program there has been some analytical progress as well, (Maldacena, Zhiboedov; Alday, Zhiboedov; Rychkov, Tan; Manashov, Strohmaier; ...);





*

Wilson-Fischer CFT's or critical $O(N)$ vector-models are known to describe many of the second-order phase transitions in $3d$ for small $N \leq 5$: wapor-liquid, Ising ($N = 1$); super-fluid transition in Helium ($N = 2$); $O(3)$-magnetic; ...

*The pictures by Matthieumarechal and SliteWrite are taken from Wikipedia

Nice Features of Wilson-Fisher CFT

- are directly related to physics;
- free of any free parameters, e.g. coupling constants etc.;
- provide infinitely many of measurable (calculable) observables — critical indices

$$\langle OO\rangle \sim \frac{1}{x^{2\Delta_O}} \qquad\qquad \Delta_O = \text{free} + \text{anomalous}$$

- are well-defined QFT's according to mathematical standards of rigour;
- no exact solution is known;

Especially Nice Features of Wilson-Fisher CFT

- can be used to define some QG — HS theories;
- the opposite argument is that there should be some HS symmetry behind (requires $y > 1998, 2002$);
- almost have ∞-dim. symmetry — HS symmetry, which is realized by

$$\partial^m J_{ma(s-1)} = \text{something}$$

$$\Delta(J_2) = 3 \text{ a must!}$$
$$\Delta(J_4) = 5.0208 \text{ while in free it is } 5.0(0)$$

As the spin increases the currents more and more look like free ones — the convexity of anomalous dimensions, (Komargodsky, Zhiboedov), $\delta J_\infty = s + 2\gamma_\phi$.

HS Breaking

One can think of two mechanisms, each can be considered on either CFT or AdS side.

3-names	CFT side	AdS side
BEH	via other single-trace operators, $\partial J = gO_1$, (Sundborg; Bianchi)	massless absorbs massive, $(s,m)\|_{m\to 0} = (s,0) \oplus (s\text{-}1, M)$ (Zinoviev; Bianchi)
GPZ	via double-, triple-trace operators, $\partial \cdot J = gJJ$	radiative corrections?

In the former case breaking cries for appending pure HS theory with some matter like fields. In the latter case the fate of gauge invariance is unclear.

$$S = \int d^dx \left[(\partial\phi)^2 + \frac{g\mu^\epsilon}{4}(\phi^2)^2 \right] .$$

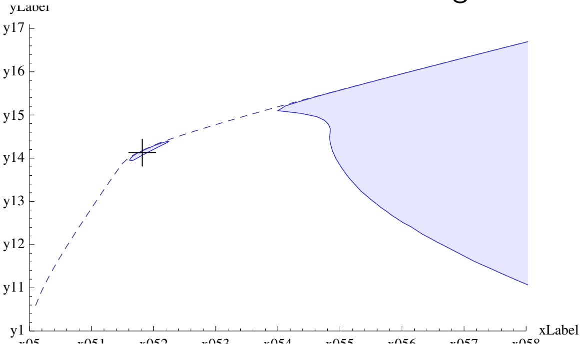
The one-loop results for the β-function and anomalous dimensions of the operators ϕ^i, $i = 1, ..., N$ and ϕ^2 are:

$$\beta = -\epsilon g + (N+8)\frac{g^2}{8\pi^2}\,, \qquad g_* = \frac{8\pi^2}{N+8}\epsilon\,,$$

$$\gamma_\phi = \frac{N+2}{4(N+8)^2}\epsilon^2\,, \qquad \Delta_\phi = \frac{d}{2} - 1 + \gamma_\phi\,,$$

$$\gamma_{\phi^2} = \frac{N+2}{N+8}\epsilon\,, \qquad \Delta_\phi = d - 2 + \gamma_{\phi^2}$$

A picture from 'Bootstrapping Mixed Correlators in the 3D Ising Model' by Filip Kos, David Poland, David Simmons-Duffin illustrates the difference between the continuum of CFT's and interesting CFT's:



$$\Box\phi = g_\star \phi^3$$

Anomalous dimensions: e.o.m insertions

The e.o.m. are still useful at the quantum level. Given

$$\langle \phi^i \phi^j \rangle = \delta^{ij} \frac{C_{\phi\phi}}{(x_{12}^2)^{\Delta}_{\phi}}$$

we can apply box twice to get, $\Delta = (d-2)/2 + \gamma$,

$$\Box_1 \Box_2 \langle \phi\phi \rangle = 4\gamma(\gamma+1)(2\gamma+d-2)(2\gamma+d)\frac{C_{\phi\phi}}{(x_{12}^2)^{\Delta_\phi+2}}$$

For the large-N vector model with $\Box\phi = \sigma\phi$:

$$C_{\sigma\sigma} g_\star^2 = 4d(d-2)\gamma_\phi$$

For the Wilson-Fisher $4-2\epsilon$ with $\Box\phi = g_\star \phi^2 \phi$:

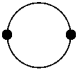
$$\gamma_\phi = \frac{(N+2)g_\star^2}{256\pi^4}$$

For the large-N vector model

$$S=\int \frac{1}{2}\partial\phi^2+\frac{1}{2}(\phi^2)\sigma+\frac{3N}{2\lambda}\sigma^2 \qquad \rightarrow \qquad \Box\phi^i=\phi^i\sigma$$

we have $\Box\phi=\sigma\phi$ and

$$\langle\phi\phi\rangle=\frac{\Gamma[d/2-1]}{2\pi^{d/2}}\frac{1}{x^{d-2}}$$

$\langle\sigma\sigma\rangle=\Big[-\frac{1}{2}$  $\Big]^{-1}=\dfrac{2^{d+2}\Gamma\left[\frac{d-1}{2}\right]\,\mathrm{Sin}\left[\pi\frac{d}{2}\right]}{\pi^{3/2}\Gamma[d/2-2]x^4}$

$\gamma_\phi=$  $=\dfrac{2\mathrm{Sin}[\pi\frac{d}{2}]\Gamma[d-2]}{\pi\Gamma[d/2+1]\Gamma[d/2-2]}$

Combining the two quantities we see that

$$C_{\sigma\sigma}=4d(d-2)\gamma_\phi$$

Anomalous dimensions: Anselmi's trick

Trivial identity — check of non-conservation

$$\langle \mathbb{D}_1 J_s(x_1) \mathbb{D}_2 J_s(x_2)\rangle = \mathbb{D}_1\mathbb{D}_2 \langle J_s(x_1) J_s(x_2)\rangle$$

can give important information provided the two sides can be computed independently. Let J be anomalous

$$\langle J_s(x_1,\eta_1) J_s(x_2,\eta_2)\rangle = \frac{C_s}{\mu^{2\gamma}(x_{12}^2)^{d+s-2+\gamma}}(P_{12})^s$$

and the non-conservation be via double-trace operators

$$K = \mathbb{D}J = g_\star JJ \qquad\qquad g_\star \sim \frac{1}{N}, \epsilon$$

The ratio gains $g_\star^2$ on the left and γ on the right

$$g_\star^2 \frac{\langle KK\rangle}{\langle JJ\rangle} \sim \gamma$$

Non-conservation: ϵ-expansion

The conserved currents are produced by Gegenbauer polynomials. More precisely $\phi\partial^s\phi$ is (Todorov et al)

$$J = (\hat{\partial}_1 + \hat{\partial}_2)^s C_s^{\frac{d-3}{2}} \left(\frac{\hat{\partial}_1 - \hat{\partial}_2}{\hat{\partial}_1 + \hat{\partial}_2}\right) \phi(x_1)\phi(x_2)\Big|_{x_1=x_2=x}$$

where $\hat{\partial}_i = \xi \cdot \partial_i$. If $\Box\phi = g_\star \phi\phi^2$ then we get

$$\partial \cdot J = g_\star(\hat{\partial}_1 + \hat{\partial}_2 + \hat{\partial}_3 + \hat{\partial}_4)^{s-1} K(u,v)\, \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\Big|_{x_i=x}$$

where the point-splitting arguments are

$$u = \frac{\hat{\partial}_1 - \hat{\partial} + \hat{\partial}_3 + \hat{\partial}_4}{\hat{\partial}_1 + \hat{\partial}_2 + \hat{\partial}_3 + \hat{\partial}_4} \qquad v = \frac{\partial_1 - \hat{\partial}_2 - \hat{\partial}_3 - \hat{\partial}_4}{\hat{\partial}_1 + \hat{\partial}_2 + \hat{\partial}_3 + \hat{\partial}_4}$$

and K is a certain combination of Gegenbauer polynomials.

For the Wilson-Fischer CFT defined by

$$S=\int \frac{1}{2}\partial\phi^2+\frac{g}{4}\mu^{2\epsilon}(\phi^2)^2 \qquad \rightarrow \qquad \Box\phi^i=g_\star\phi^i\phi^2$$

anomalous dim. of non-singlet currents $\phi^i\partial^s\phi^j$ are

$$\gamma^{\Box\Box}=2\gamma_\phi\left(1-\frac{2(N+6)}{(N+2)s(s+1)}\right) \quad \gamma^{\substack{\Box\\\Box}}=2\gamma_\phi\left(1-\frac{2}{s(s+1)}\right)$$

and anomalous dimensions the singlet currents $\phi^i\partial^s\phi_i$ are

$$\gamma=2\gamma_\phi\left(1-\frac{6}{s(s+1)}\right)$$

which vanishes for the stress-tensor $s=2$. This requires one-loop, see Wilson, Kogut

Non-conservation: ϵ-expansion

Large-N expansion is based on

$$S = \int \frac{1}{2}\partial\phi^2 + \frac{1}{2}(\phi^2)\sigma + \frac{3N}{2\lambda}\sigma^2 \qquad \to \qquad \Box\phi^i = \phi^i\sigma$$

Using $\Box\phi = g_\star\phi\sigma$ we get instead

$$\partial\cdot J = g_\star(\hat{\partial}_1 + \hat{\partial}_2 + \hat{\partial}_3)^{s-1}K(u,v)\,\phi(x_1)\phi(x_2)\sigma(x_3)\Big|_{x_i=x}$$

with similar point-splitting arguments.

We can decompose it into irreducibles to make JJ manifest:

$$\mathbb{D}J_s = g_\star \sum_{a+c<s} C^s_{a,c}\,\partial^a J_{s-1-a-c}\,\partial^c\sigma\,.$$

For example, (Maldacena, Zhiboedov; Giombi et al),
$\mathbb{D}J_4 \sim J_2\,\partial\sigma - \frac{2}{5}\partial J_2\,\sigma$

Large-N Vector-Model

Klebanov-Polyakov duality is based on the large-N of

$$S = \int \frac{1}{2}\partial\phi^2 + \frac{1}{2}(\phi^2)\sigma + \frac{3N}{2\lambda}\sigma^2 \qquad \rightarrow \qquad \Box\phi^i = \phi^i\sigma$$

The anomalous dimensions of the non-singlet currents are

$$\gamma_s = \frac{8(s-1)(d+s-2)}{(d+2s-4)(d+2s-2)}$$

which agrees with (Lang, Ruhl) and vanishes for $s=1$ since it is a protected current generating global $O(N)$-symmetry. That of the singlet currents is a bit trickier and in $3d$ is:

$$\gamma_s = 4\gamma_\phi \frac{(s-2)}{(2s-1)}$$

Scalar QCD or $6d$ ϕ^3

Toy model of QCD: asymptotically free theory in $6d$

$$S = \int \frac{1}{2}\partial\phi^2 + \partial\bar{\psi}\partial\psi + g\bar{\psi}\phi\psi$$

The anomalous dimensions of the non-singlet currents are

$$\gamma_1 = \frac{(s-1)(s+4)}{24(s+1)(s+2)} = \frac{1}{4}\left(\frac{1}{6} - \frac{1}{(s+1)(s+2)}\right)$$

which agrees with (Kubota; Radyshkin; Belitsky et al) and vanishes for $s=1$ since it is a protected current that generates global $O(N)$-symmetry.

Anomalous dimensions to the one-loop order are ratios:

$$c_s\gamma_s = \frac{\langle K|K\rangle}{\langle J|J\rangle} = \frac{\langle J\otimes J|J\otimes J\rangle}{\langle J|J\rangle}$$

The two-point functions are related to the trace on the HS algebra. The non-conservation operator $\mathbb{D}J = K = JJ$ belongs to the tensor square of the HS algebra representation $J \sim \phi\otimes\phi$. Therefore, to find K we need to implicitly know the Clebsh-Gordon coefficients of the HS algebra.

The kinematical factor c_s comes from the HS algebra of generalized free fields (Eastwood; Alkalaev, Grigoriev, E.S.) at $\Delta = (d-2)/2 + \epsilon$ when the ideal is formed:

$$U\left(\ydiagram{1,1}\right) \Big/ \left[\ydiagram{1,1,1,1} \oplus (C_2 - C_2(\Delta))\right] \qquad\qquad I \sim \ydiagram{2}$$

HS algebra interpretation

In free CFT's we have the following chain

$$\partial \cdot J_s = 0 \quad \Longrightarrow \quad Q_s = \int J_s \quad \Longrightarrow \quad hs - \text{algebra}$$

and correlators are HS algebra invariants.

In the reality the conservation of the HS currents is broken by themselves. What is the right algebraic structure that captures this phenomenon?

$$\partial \cdot J_s = JJ \quad \Longrightarrow \quad Q_s(?) \quad \Longrightarrow \quad ??$$

Wilson-Fisher in $4-\epsilon$ should be dual to HS theory in $AdS_{5-\epsilon}$.

The mass shift of the HS fields in $AdS_{5-\epsilon}$ HS theory has to be

$$\boxed{\delta m^2 = -2(s-2)\epsilon + 2\epsilon^2\gamma_\phi \left(1-\frac{6}{s(s+1)}\right)}$$

To avoid problems with gauge invariance the massless spin-s field can recombine with a massive spin-$(s-1)$ field to form a massive spin-s field, (Bianchi). At the free level the action proposed by Zinoviev does the job

$$S = S_{s,m=0} + \delta m^2 \int \Phi_s \partial \Phi_{s-1} + S_{s-1,M}$$

In unfolding: $dC = [\omega, C]$ for C in $hs \times hs \times ...$.

In the $1/N$-expansion the situation is analogous, with $\delta m^2 \sim (s-2)/N$, Ruhl.

Conclusions

Unbroken HS symmetry is very rigid, unique and leads to free fields. Everything is just HS algebra rep. theory.

There are several ways to break HS symmetry as to get access to interesting CFT's. The most promising is the quantum breaking via double-trace operators.

The study of quantum equations of motion manifesting the breaking of the HS symmetry allows to determine all one-loop anomalous dimensions. The method works nicely for all $CFT's$ in diverse dimensions that were treated by distinct methods. One-loop again can be explained by HS algebras. What is the right structure?