Black Holes, Higher Derivatives and Stability

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Consider the gravitational action

$$I = \int d^4x\sqrt{-g}(\gamma R - \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2)\,.$$

The field equations following from this higher-derivative action are

$$\begin{aligned} H_{\mu\nu} \quad = \quad & \gamma\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{2}{3}(\alpha - 3\beta)\nabla_\mu\nabla_\nu R - 2\alpha\Box R_{\mu\nu} \\ & + \frac{1}{3}(\alpha + 6\beta)\,g_{\mu\nu}\Box R - 4\alpha R^{\eta\lambda}R_{\mu\eta\nu\lambda} + 2\left(\beta + \frac{2}{3}\alpha\right)RR_{\mu\nu} \\ & + \frac{1}{2}g_{\mu\nu}\left(2\alpha R^{\eta\lambda}R_{\eta\lambda} - \left(\beta + \frac{2}{3}\alpha\right)R^2\right) = \frac{1}{2}T_{\mu\nu} \end{aligned}$$

Nonlinear field equations for spherical symmetry

Use Schwarzschild coordinates
$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$

The first equation contains the third-order derivative $B^{(3)} = B'''$

$$
\begin{aligned}
24r^4A^3B^4H_{rr} = {} & 8r^3A^2B^2B^{(3)}\left(r(\alpha-3\beta)B' - 2(\alpha+6\beta)B\right) \\
& - 4r^2AB^2A''\left(r^2(\alpha-3\beta)B'^2 - 4r(\alpha+6\beta)BB' + 4(\alpha-12\beta)B^2\right) \\
& - 4r^4(\alpha-3\beta)A^2B^2B''^2 \\
& - 4r^2ABB''\Big(2rBA'\left(r(\alpha-3\beta)B' - 2(\alpha+6\beta)B\right) \\
& \quad + A\left(3r^2(\alpha-3\beta)B'^2 - 12r(\alpha+3\beta)BB' + 8(\alpha+6\beta)B^2\right)\Big) \\
& + 7r^2B^2A'^2\left(r^2(\alpha-3\beta)B'^2 - 4r(\alpha+6\beta)BB' + 4(\alpha-12\beta)B^2\right) \\
& + 2r^2ABA'B'\left(3r^2(\alpha-3\beta)B'^2 - 4r(2\alpha+3\beta)BB' + 4(\alpha+24\beta)B^2\right) \\
& + 24A^3B^3\left(\gamma r^3B' + B\left(\gamma r^2 - 12\beta\right)\right) \\
& + A^2\Big(7r^4(\alpha-3\beta)B'^4 - 4r^3(5\alpha+12\beta)BB'^3 \\
& \quad - 4r^2(\alpha-48\beta)B^2B'^2 + 32r(\alpha+6\beta)B^3B' - 16(\alpha-21\beta)B^4\Big) \\
& + 8A^4B^4\left(2\alpha - 6\beta - 3\gamma r^2\right)
\end{aligned}
$$

The second equation contains the third-order derivative $A^{(3)} = A'''$:

$$
\begin{aligned}
&2r^4A^5B^2\left(\alpha rB' - 3\beta rB' - 2\alpha B - 12\beta B\right)^2 \left(H_{tt} - X(r)H_{rr} - Y(r)\partial_r H_{rr}\right) = \\
&\quad 72\alpha\beta r^3 A^2 A^{(3)} B^4 \left(r(\alpha-3\beta)B' - 2(\alpha+6\beta)B\right) \\
&\quad +36\alpha\beta r^2 A B^3 A''\Big(13rBA'\left(2(\alpha+6\beta)B - r(\alpha-3\beta)B'\right) \\
&\qquad -2A(-r^2(\alpha-3\beta)B'^2 + r(\alpha+6\beta)BB' + 2(\alpha+6\beta)B^2)\Big) \\
&\quad +12\beta r^4(\alpha-3\beta)A^3B^2B''^2\left((\alpha+6\beta)B - r(\alpha-3\beta)B'\right) \\
&\quad +4r^3A^2BB''\Big(3\beta BA'\left(r^2(\alpha-3\beta)^2B'^2 + r\left(\alpha^2-15\alpha\beta+36\beta^2\right)BB' - 6\alpha(\alpha+6\beta)B^2\right) \\
&\qquad -3\beta AB'\left(-r^2(\alpha-3\beta)^2B'^2 - 6\alpha r(\alpha-3\beta)BB' + 2\left(7\alpha^2+48\alpha\beta+36\beta^2\right)B^2\right) \\
&\qquad +\gamma(-r)(\alpha-3\beta)A^2B^2\left(2(\alpha+6\beta)B - r(\alpha-3\beta)B'\right)\Big) \\
&\quad +504\alpha\beta r^3B^4A'^3\left(r(\alpha-3\beta)B' - 2(\alpha+6\beta)B\right) \\
&\quad -3\beta r^2AB^2A'^2\Big(r^3(\alpha-3\beta)^2B'^3 + 3r^2\left(17\alpha^2-57\alpha\beta+18\beta^2\right)BB'^2 \\
&\qquad -60\alpha r(\alpha+6\beta)B^2B' - 4\left(23\alpha^2+150\alpha\beta+72\beta^2\right)B^3\Big) \\
&\quad -6\beta rA^2BA'\Big(r^4(\alpha-3\beta)^2B'^4 + r^3\left(11\alpha^2-39\alpha\beta+18\beta^2\right)BB'^3 - 4r^2\left(8\alpha^2+51\alpha\beta+18\beta^2\right)B^2B'^2 \\
&\qquad +4r\left(11\alpha^2-12\alpha\beta+18\beta^2\right)B^3B' - 16\left(4\alpha^2+21\alpha\beta-18\beta^2\right)B^4\Big) \\
&\quad +A^3\Big(-4r(\alpha-3\beta)B^4B'\left(12\beta(5\alpha+3\beta) + r(\alpha-3\beta)A'\left(\gamma r^2-12\beta\right)\right) \\
&\qquad -2r^2B^3B'^2\left(6\beta\left(\alpha^2+66\alpha\beta+36\beta^2\right) + \gamma r^3(\alpha-3\beta)^2A'\right) \\
&\qquad -8(\alpha+6\beta)B^5\left(-6\beta(5\alpha+3\beta) - rA'\left(2\alpha\left(\gamma r^2-6\beta\right) + 3\beta\left(12\beta+\gamma r^2\right)\right)\right) \\
&\qquad -3\beta r^5(\alpha-3\beta)^2B'^5 + 3\beta r^4\left(-19\alpha^2+51\alpha\beta+18\beta^2\right)BB'^4 + 12\beta r^3\left(13\alpha^2+84\alpha\beta+36\beta^2\right)B^2B'^3\Big) \\
&\quad -8A^5B^4\Big(r(\alpha-3\beta)B'\left(\alpha\left(\gamma r^2-6\beta\right) + 6\beta\left(3\beta+\gamma r^2\right)\right) \\
&\qquad +(\alpha+6\beta)B\left(\alpha\left(6\beta-2\gamma r^2\right) - 3\beta\left(6\beta+\gamma r^2\right)\right)\Big) \\
&\quad -2A^4B^2\Big(\gamma r^5(\alpha-3\beta)^2B'^3 - 6r^2(\alpha-3\beta)BB'^2\left(\alpha\left(\gamma r^2-4\beta\right) + 3\beta\left(4\beta+\gamma r^2\right)\right) \\
&\quad +4r(\alpha-3\beta)B^2B'\left(\alpha\left(\gamma r^2-24\beta\right) + 6\beta\left(\gamma r^2-6\beta\right)\right) + 4\left(2\alpha^2+15\alpha\beta+18\beta^2\right)B^3\left(12\beta+\gamma r^2\right)\Big).
\end{aligned}
$$

Separation of modes in the linearized theory

Solving the full nonlinear field equations is clearly a challenge. One can make initial progress by restricting the metric to infinitesimal fluctuations about flat space, defining $h_{\mu\nu} = \kappa^{-1}(g_{\mu\nu} - \eta_{\mu\nu})$ and then restricting attention to field equations linearized in $h_{\mu\nu}$, or equivalently by restricting attention to quadratic terms in $h_{\mu\nu}$ in the action.

The action then becomes

$$I_{\rm Lin} = \int d^4x \{ -\frac{1}{4} h^{\mu\nu}(2\alpha\Box - \gamma)\Box P^{(2)}_{\mu\nu\rho\sigma} h^{\rho\sigma} + \frac{1}{2} h^{\mu\nu}[6\beta\Box - \gamma]\Box P^{(0;{\rm s})}_{\mu\nu\rho\sigma} h^{\rho\sigma} \} ;$$

$$P^{(2)}_{\mu\nu\rho\sigma} = \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - P^{(0;{\rm s})}_{\mu\nu\rho\sigma}$$

$$P^{(0;{\rm s})}_{\mu\nu\rho\sigma} = \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \qquad \theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu} \qquad \omega_{\mu\nu} = \partial_\mu\partial_\nu/\Box \ ,$$

where the indices are lowered and raised with the background metric $\eta_{\mu\nu}$.

From this linearized action one deduces the dynamical content of the linearized theory:

- *positive-energy* massless spin-two
- *negative-energy* massive spin-two with mass $m_2 = \gamma^{\frac{1}{2}}(2\alpha)^{-\frac{1}{2}}$
- *positive-energy* massive spin-zero with mass $m_0 = \gamma^{\frac{1}{2}}(6\beta)^{-\frac{1}{2}}$

  K.S.S. 1978

Static and spherically symmetric solutions

Now consider spherically symmetric gravitational solutions in the linearised limit of the higher-curvature theory. In the linearized theory, one finds the following general solution to the source-free field equations $H^L_{\mu\nu} = 0$, in which $C, C^{2,0}, C^{2,+}, C^{2,-}, C^{0,+}, C^{0,-}$ are integration constants:

$$A(r) =$$
$$1 - \frac{C^{20}}{r} - C^{2+}\frac{e^{m_2 r}}{2r} - C^{2-}\frac{e^{-m_2 r}}{2r} + C^{0+}\frac{e^{m_0 r}}{r} + C^{0-}\frac{e^{-m_0 r}}{r}$$
$$+\frac{1}{2}C^{2+}m_2 e^{m_2 r} - \frac{1}{2}C^{2-}m_2 e^{-m_2 r} - C^{0+}m_0 e^{m_0 r} + C^{0-}m_0 e^{-m_0 r}$$

$$B(r) =$$
$$C + \frac{C^{20}}{r} + C^{2+}\frac{e^{m_2 r}}{r} + C^{2-}\frac{e^{-m_2 r}}{r} + C^{0+}\frac{e^{m_0 r}}{r} + C^{0-}\frac{e^{-m_0 r}}{r}$$

- As one might expect from the dynamics of the linearized theory, the general static, spherically symmetric solution is a combination of a massless Newtonian $1/r$ potential plus rising and falling Yukawa potentials arising in both the spin-two and spin-zero sectors.
- When coupling to non-gravitational matter fields is made via standard $h^{\mu\nu}T_{\mu\nu}$ minimal coupling, one gets values for the integration constants from the specific form of the source stress tensor. Requiring asymptotic flatness and coupling to a point-source positive-energy matter delta function $T_{\mu\nu} = \delta^0_\mu \delta^0_\nu M \delta^3(\vec{x})$, for example, one finds

$$\begin{aligned} A(r) &= 1 + \frac{\kappa^2 M}{8\pi\gamma r} - \frac{\kappa^2 M(1+m_2 r)}{12\pi\gamma}\frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M(1+m_0 r)}{48\pi\gamma}\frac{e^{-m_0 r}}{r} \\ B(r) &= 1 - \frac{\kappa^2 M}{8\pi\gamma r} + \frac{\kappa^2 M}{6\pi\gamma}\frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M}{24\pi\gamma}\frac{e^{-m_0 r}}{r} \end{aligned}$$

with specific combinations of the Newtonian $1/r$ and falling Yukawa potential corrections arising from the spin-two and spin-zero sectors.

Note that in the Einstein-plus-quadratic-curvature theory, there is *no Birkhoff theorem*. For example, in the linearized theory, coupling to the stress tensor for an extended source like a perfect fluid with pressure P constrained within a radius ℓ by an elastic membrane,

$$T_{\mu\nu} = \mathrm{diag}[P, [P-\tfrac{1}{2}\ell\delta(r-\ell)]r^2, [P-\tfrac{1}{2}\ell\delta(r-\ell)]r^2\sin^2\theta, 3M(4\pi\ell^3)^{-1}]\,,$$

one finds for the external $B(r)$ function

$$\begin{aligned} B(r) = 1 - \frac{\kappa^2 M}{8\pi\gamma r} + \frac{\kappa^2 e^{-m_2 r}}{\gamma r}\Bigg\{\frac{M}{2\pi\ell^3}\left[\frac{\ell\cosh(m_2\ell)}{m_2^2} - \frac{\sinh(m_2\ell)}{m_2^3}\right] \\ -P\left[\frac{\sinh(m_2\ell)}{m_2^3} - \frac{\ell\cosh(m_2\ell)}{m_2^2} + \frac{\ell^2\sinh(m_2\ell)}{3m_2}\right]\Bigg\} \\ -\frac{\kappa^2 e^{-m_0 r}}{2\gamma r}\Bigg\{\frac{M}{4\pi\ell^3}\left[\frac{\ell\cosh(m_0\ell)}{m_0^2} - \frac{\sinh(m_0\ell)}{m_0^3}\right] \\ -P\left[\frac{\sinh(m_0\ell)}{m_0^3} - \frac{\ell\cosh(m_0\ell)}{m_0^2} + \frac{\ell^2\sinh(m_0\ell)}{3m_0}\right]\Bigg\} \end{aligned}$$

which limits to the point-source result as $\ell \to 0$.

Frobenius Asymptotic Analysis

Asymptotic analysis of the field equations near the origin leads to study of the *indicial equations* for behavior as $r \to 0$. K.S.S. 1978
Let

$$\begin{aligned} A(r) &= a_s r^s + a_{s+1} r^{s+1} + a_{s+2} r^{s+2} + \cdots \\ B(r) &= b_t r^t + b_{t+1} r^{t+1} + b_{t+2} r^{t+2} + \cdots \end{aligned}$$

and analyze the conditions necessary for the lowest-order terms in r of the field equations $H_{\mu\nu} = 0$ to be satisfied. This gives the following results, for the general $\alpha,\ \beta$ theory:

$$\begin{aligned} (s,t) &= (1,-1) && \text{with 5 free parameters} \\ (s,t) &= (0,0) && \text{with 3 free parameters} \\ (s,t) &= (2,2) && \text{with 6 free parameters} \end{aligned}$$

Lü, Perkins, Pope & K.S.S., 1508.00010

Now suppose one puts an "egg-shell" δ-function source at some small distance ϵ from the origin. Consider solving these sourced equations, similarly to the linearized theory analysis. Inside the shell, the solution can only be of the (0,0) nonsingular type, which needs no source. Suppose that outside one has a solution that would be of (2,2) type if one continued it all the way in to $r = 0$.

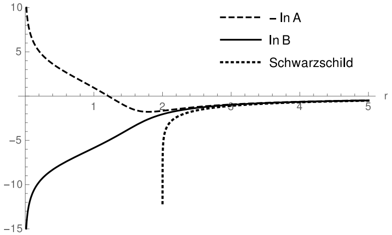
Count parameters: 3 inside + 6 outside = 9 initially. However, there are 6 continuity and 'jump' conditions coming from the field equations. So one really has 9-6=3 parameters still free. These 3 so-far unfixed parameters are just what is needed for 2 boundary conditions at infinity, to eliminate the rising exponential solutions, plus the 'trivial' parameter that is fixed by requiring $g_{00} \to -1$ as $r \to \infty$.

Conclusion: the exterior (2,2) solution works well with a source in the full non-linear theory. Exterior (1,-1) and (0,0) solutions would, however, be *overdetermined*. So coupling to a standard positive-energy source works only in the (2,2) family.

(2,2) solutions without horizons

For asymptotically flat solutions with nonzero spin-two Yukawa coefficient $C^{2-} \neq 0$, one finds numerical solutions that can continue on in to mesh with the (2,2) family obtained from Frobenius asymptotic analysis around the origin. Such solutions have no horizon; numerical solutions have been found in the $m_2 = m_0$ theory B. Holdom, Phys.Rev. D66 (2002) 084010 and in the $R + C^2$ theory

Lü, Perkins, Pope & K.S.S., 1508.00010

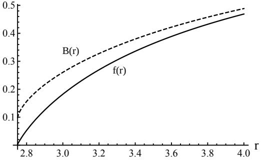


Horizonless solution in $R + C^2$ theory, behaving as r^2 in both $A(r)$ and $B(r)$ as $r \to 0$.

- *There is no horizon* in this set of minimally-coupled, Yukawa-corrected solutions. Solutions asymptotically approach the Schwarzschild solution for large r, but differ strikingly in what would have been the inner-horizon region.
- This is in accord with generic conclusions from the parameter count for solutions. For a generic $R - C^2 + R^2$ theory solution, there will be both spin-two and spin-zero falling Yukawa terms as one approaches spatial infinity. Together with the trivial time-rescaling parameter and the mass M, this makes four welcome parameters. One then needs two more solution parameters to ensure cancellation of the unwelcome rising spin-two and spin-zero exponential terms.
- Although there is a curvature singularity at the origin in the $(2, 2)$ class of solutions (*e.g.* for this class, one has $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 20a_2^{-2}r^{-8} + \cdots$), this is a *timelike* singularity, unlike the *spacelike* singularity of the Schwarzschild solution.

Wormholes

Another solution type found numerically has the character of a "wormhole". Such solutions can have either sign of $M \sim -C^{20}$ and either sign of the falling Yukawa coefficient C^{2-}. As an example, one finds a solution with $M < 0$ in the $R - C^2$ theory



In this solution, $f(r) = 1/A(r)$ reaches zero at a point where $B(r) = a_0^2 > 0$. Making a coordinate change $r - r_0 = \frac{1}{4}\rho^2$, one then has

$$ds^2 = -(a_0^2 + \tfrac{1}{4}B'(r_0)\rho^2)dt^2 + \frac{d\rho^2}{f'(r_0)} + (r_0^2 + \tfrac{1}{2}r_0\rho^2)d\Omega^2$$

which is $\mathbb{Z}_2$ symmetric in ρ and can be interpreted as a "wormhole", with the $r < r_0$ region excluded from spacetime.

No-hair Theorems and Horizons

W. Nelson, Phys.Rev. D82 (2010) 104026; arXiv:1010.3986; H. Lü, A. Perkins, C.N. Pope & K.S.S., 1508.00010

- For $\beta > 0$ (*i.e.* for non-tachyonic $m_0^2 > 0$), take the trace of the $H_{\mu\nu} = 0$ field equation: $\left(\Box - \frac{\gamma}{6\beta}\right) R = 0$. Then multiply by $\lambda^{\frac{1}{2}} R$ and integrate with $\int \sqrt{h}$ over a 3D spatial slice at a fixed time, on which h_{ab} is the 3D metric and $\lambda = -t^a t^b g_{ab}$ is the norm^2 of the timelike Killing vector t^a orthogonal to the slice. Integrating by parts, one obtains

$$\int d^3x \sqrt{h}[D^a(\lambda^{\frac{1}{2}} R D_a R) - \lambda^{\frac{1}{2}} (D^a R)(D_a R) - m_0^2 \lambda^{\frac{1}{2}} R^2] = 0$$

  where D_a is a 3D covariant derivative on the spatial slice.

  From this, provided the boundary term arising from the total derivative gives a zero contribution, and for $m_0^2 > 0$, one learns $R = 0$. The boundary at spatial infinity gives a vanishing contribution provided $R \to 0$ as $r \to \infty$.
- The inner boundary at a horizon null-surface will give a zero contribution since $\lambda = 0$ there.

Consequently, for asymptotically flat solutions with a horizon, one concludes that one must have $R = 0$. This already excludes the possibility of the scalar m_0 Yukawa correction found in the limit as $r \to \infty$. So, for solutions that do have such a scalar Yukawa correction to the classic GR behavior, one directly concludes: *there can be no horizon.*

What about the non-trace part of the field equation and the spin-two m_2 Yukawa corrections? Nelson's paper would have allowed one to make a similar conclusion for the rest of $R_{\mu\nu}$. Unfortunately, detailed analysis of his paper shows that it has a fundamental flaw: instead of a sum of squares of the same sign, one gets squares of opposite signs. Lü, Perkins, Pope & K.S.S., 1508.00010

If one assumes the existence of a horizon and assumes also asymptotic flatness at infinity, one obtains $R = 0$ as above. The field equations then become identical to those in the special $\beta = 0$ case, *i.e.* with just a $(\text{Weyl})^2$ term and no R^2 term in the action.

Counting parameters in an expansion around the horizon, subject to the $R = 0$ condition, one finds just 3 free parameters. This is the same count as in the (1,-1) family of the expansion around the origin when subjected to the $R = 0$ condition. So asymptotically flat solutions with a horizon must belong uniquely to the (1-1) family, which contains the Schwarzschild solution itself. The Schwarzschild solution is characterized by two parameters: the mass M of the black hole, plus the trivial g_{00} normalization at infinity. So in the higher-derivative theory, there is just one "non-Schwarzschild" (1,-1) parameter.

Away from Schwarzschild in the (1,-1) family

Lü, Perkins, Pope & K.S.S., 1508.00010

Considering variation of this "non-Schwarzschild" parameter away from the Schwarzschild value, it is clear that changing it has to do something to the solution at infinity. For a solution assumed to have a horizon, and holding $R = 0$, the only thing that can happen initially is that the rising exponential is turned on, *i.e.* asymptotic flatness is lost. So, for asymptotically flat solutions with a horizon *in the vicinity of the Schwarzschild solution*, the only spherically symmetric static solution is Schwarzschild itself.

This conclusion is formalized by considering infinitesimal variations of a solution away from Schwarzschild and proving a no-hair theorem for the *linearized* equation in the variation. This can successfully be done for coefficients α that are not too large (*i.e.* for spin-two masses m_2 that are not too small). One concludes that the Schwarzschild black hole is at least in general *isolated* as an asymptotically flat solution with a horizon.

Non-Schwarzschild Black Holes

Lü, Perkins, Pope & K.S.S., PRL 114, 171601 (2015); arXiv 1502.01028

Now the question arises: what happens when one moves a finite distance away from Schwarzschild in terms of the (1,-1) non-Schwarzschild parameter? Does the loss of asymptotic flatness persist, or does something else happen, with solutions arising that cannot be treated by a linearized analysis in deviation from Schwarzchild?

This can be answered numerically. In consequence of the trace no-hair theorem, the assumption of a horizon together with asymptotic flatness requires $R = 0$ for the solution, so the calculations can effectively be done in the $R - C^2$ theory with $\beta = 0$, in which the field equations, thankfully, can be reduced to a system of two second-order equations.

The study of non-Schwarzschild solutions is more easily carried out with a metric parametrization

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

i.e. by letting $A(r) = 1/f(r)$.

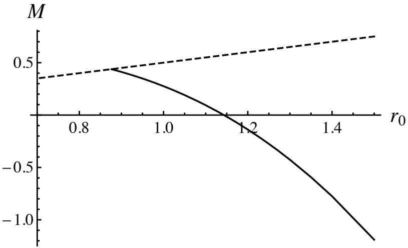
For $B(r)$ vanishing linearly in $r - r_0$ for some r_0, analysis of the field equations shows that one must then also have $f(r)$ similarly linearly vanishing at r_0, and accordingly one has a horizon. One can thus make near-horizon expansions

$$\begin{aligned} B(r) &= c\left[(r-r_0) + h_2\,(r-r_0)^2 + h_3\,(r-r_0)^3 + \cdots\right] \\ f(r) &= f_1\,(r-r_0) + f_2\,(r-r_0)^2 + f_3\,(r-r_0)^3 + \cdots \end{aligned}$$

and the parameters h_i and f_i for $i \geq 2$ can then be solved-for in terms of r_0 and f_1. For the Schwarzschild solution, one has $f_1 = 1/r_0$, so it is convenient to parametrize the deviation from Schwarzschild using a non-Schwarzschild parameter δ with

$$f_1 = \frac{1+\delta}{r_0}\,.$$

The task then becomes that of finding values of $\delta \neq 0$ for which the generic rising exponential behavior as $r \to \infty$ is suppressed. What one finds is that there do indeed exist asymptotically flat non-Schwarzschild black holes provided the horizon radius r_0 exceeds a certain minimum value $r_0^{\rm min}$. For $\alpha = \frac{1}{2}$, one finds the following phases of black holes:



Black-hole masses as a function of horizon radius r_0, with a branch point at $r_0^{\rm min} \simeq 0.876$. The dashed line denotes Schwarzschild black holes and the solid line denotes non-Schwarzschild black holes.

The Lichnerowicz Operator

Now let us study in some more detail the point where the new black hole family branches off from the classic Schwarzschild solution family. We can study solutions in the vicinity of the Schwarzschild family by looking at infinitesimal variations of the higher-derivative equations of motion around a Ricci-flat background. For the $\delta R_{\mu\nu}$ variation of the Ricci tensor away from a background with $R_{\mu\nu} = 0$ one obtains

$$\gamma(\delta R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\,\delta R) + 2(\beta - \tfrac{1}{3}\alpha)(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\delta R$$
$$-2\alpha\Box\,(\delta R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}\,\delta R) - 4\alpha\, R_{\mu\rho\nu\sigma}\,\delta R^{\rho\sigma} = 0\ .$$

Restricting attention to asymptotically flat solutions with horizons, however, we know from the trace no-hair theorem that $R = 0$ so $\delta R = 0$ and the $\delta R_{\mu\nu}$ equation simplifies, upon recalling that $m_2^2 = \frac{\gamma}{2\alpha}$, to

$$\left(\Delta_L + m_2^2\right) \delta R_{\mu\nu} = 0 \ ,$$

where the Lichnerowicz operator is given by

$$\Delta_L \, \delta R_{\mu\nu} \equiv -\Box \delta R_{\mu\nu} - 2 R_{\mu\rho\nu\sigma} \, \delta R^{\rho\sigma} \ .$$

Restricting attention to the $m_2^2 > 0$ nontachyonic case, one sees that black hole solutions deviating from Schwarzschild must have a $\lambda = -m_2^2$ negative Lichnerowicz eigenvalue for $\delta R_{\mu\nu}$.

The Gross-Perry-Yaffe eigenvalue

In a study of the instability of the Euclideanised Schwarzschild solution in Einstein theory, Gross, Perry and Yaffe Phys.Rev. D25 (1982), 330 found that there is just one normalisable negative-eigenvalue mode of the Lichnerowicz operator for deviations from the Schwarzschild solution. For a Schwarzschild solution of mass M, it is

$$\begin{aligned} \lambda &\simeq -0.19 M^{-2} \\ \textit{i.e.} \quad m_2 M &\simeq 0.44 \end{aligned}$$

- Comparing with the numerical results for the new black hole solutions of the higher-derivative gravity theory, this corresponds nicely with the point where the new black hole family branches off from the Schwarzschild family. This link to static new solutions was actually noted already by Brian Whitt in an early study of black hole stability in the higher-derivative theory Phys.Rev. D32 (1985), 379 , but we shall see that his conclusions about time-dependent solutions need correcting.

Analogous negative Lichnerowicz eigenvalues exist in other space-time dimensions D as well, provided that one considers only theories that admit Ricci-flat solutions (*i.e.* theories without a $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ term, since for $D \neq 4$ this is not related by the Gauss-Bonnet invariant to $(\mathrm{Ricci})^2$ and R^2 terms). Thus one considers a dimension D theory with Lagrangian

$$\mathcal{L} = \sqrt{-g}\,(R + \xi\, R^{\mu\nu} R_{\mu\nu} + \tilde{\beta}\, R^2)\ .$$

(relation to earlier $D = 4$ notation: $\gamma = 1$, $\xi = -2\alpha$, $\tilde{\beta} = \beta + \frac{2}{3}\alpha$).

Noting that Lichnerowicz eigenvalues scale like $1/r_0^2$ and that $M = 1$ for Gross, Perry and Yaffe corresponds to $r_0 = 2$, one has for Schwarzschild solutions of unit radius $r_0 = 1$ in dimensions D the eigenvalues Lü,Perkins,Pope & K.S.S. in preparation

$$\begin{aligned} D = 4 : &\qquad \lambda_1 \approx -0.7677 \\ D = 5 : &\qquad \lambda_1 \approx -1.610 \\ D = 6 : &\qquad \lambda_1 \approx -2.499 \\ D = 7 : &\qquad \lambda_1 \approx -3.417 \\ D = 8 : &\qquad \lambda_1 \approx -4.356 \\ D = 9 : &\qquad \lambda_1 \approx -5.309 \\ D = 10 : &\qquad \lambda_1 \approx -6.272 \\ D = 11 : &\qquad \lambda_1 \approx -7.242 \ , \end{aligned}$$

which one sees includes the $D = 4$ result for $M = 1 \leftrightarrow r_0 = 2$, *i.e.* $\lambda = \lambda_2 \approx -0.7677/4 = -0.1919$. These results imply that, in dimension D higher-derivative gravity theories that admit a standard Schwarzschild black-hole solution, there should also be branches of non-Schwarzschild black-hole solutions.

Time Dependent Solutions and Stability

Now consider time-dependent perturbations $\delta R_{\mu\nu}$ away from a Schwarzschild solution in order to search for possible instabilities. For this one needs to analyse the Lichnerowicz condition $(\Delta_L + m_2^2)\,\delta R_{\mu\nu} = 0$ for time-dependent solutions. For asymptotically flat solutions with a horizon, we still have the $R = 0$ consequence of the trace no-hair theorem, so $\delta R = 0$. Then from the Bianchi identity $\nabla^\mu R_{\mu\nu} = \frac{1}{2}\nabla_\nu R$ we obtain $\nabla^\mu \delta R_{\mu\nu} = 0$, so $\delta R_{\mu\nu}$ must be a "TT" quantity.

The "TT" condition for $\delta R_{\mu\nu}$ already indicates a similarity to the situation that obtains in Pauli-Fierz theory, where the linearised field equations for a massive spin-two field $\psi_{\mu\nu}$ imply $\partial^\mu \psi_{\mu\nu} = \psi^\nu{}_\nu = 0$.

Zerilli-Schrödinger Form

Lü, Perkins, Pope & K.S.S., in preparation

Start from an ansatz for spherically-symmetric time-dependent TT modes

$$\psi_{00} = h\,\psi_0(r)\,e^{\nu t}\,, \qquad \psi_{01} = \chi(r)\,e^{\nu t}$$
$$\psi_{11} = h^{-1}\,\psi_1(r)\,e^{\nu t}\,, \qquad \psi_{ij} = r^2\,\bar\psi(r)\,\gamma_{ij}\,e^{\nu t}$$

The TT conditions imply three equations which can be solved for ψ_0, $\bar\psi$ and χ'. The Lichnerowicz eigenvalue equation implies two two-derivative equations for the (01) and (00) components. Inserting the results for ψ_0, $\bar\psi$ and χ' from the TT conditions, one can solve for the undifferentiated χ and then obtain a second-order equation purely for ψ_1.

Following Zerilli's treatment in Einstein theory PRL 24 (1970), 737 , one next introduces a new variable $\phi(r)$ defined by

$$\phi(r) = \nu^{-1}\,u(r)\,\chi(r) + v(r)\,\bar\psi(r)$$

where $u(r)$ and $v(r)$ do not depend on the imaginary frequency ν.

In order to obtain a Schrödinger form for the Lichnerowicz eigenvalue equation, one requires that in terms of the "tortoise" coordinate (ranging from $-\infty$ at the horizon to $+\infty$ at spatial infinity)

$$r_* = \int^r \frac{dr'}{h(r')}$$

the function ϕ should satisfy an equation of the form

$$\frac{d^2\phi}{dr_*^2} = W(r)\,\phi$$

in which $W(r)$ should be of the form $W(r) = \nu^2 + V(r)$ where $V(r)$ does not depend on the frequency ν.

Picking a unit Schwarzschild radius $r_0 = 1$ for $h(r) = 1 - \frac{1}{r^{D-3}}$ in spacetime dimension D, one obtains a Schrödinger-form equation

$$-\frac{d^2\phi}{dr_*^2} + \left[\nu^2 + V(r)\right]\phi = 0$$

in which the potential $V(r)$ is given by

$$V(r) = -\frac{h(r)}{r^{D-1}\,[\frac{1}{2}(D-2)(D-3) - \lambda\, r^{D-1}]^2}\, Y(r)$$

where

$$\begin{aligned} Y(r) \;&=\; -\tfrac{1}{16}(D-2)^3\,(D-3)^2\,[(D-2)+(D-4)r^{D-3}] \\ &\quad +\tfrac{1}{4}(D-2)(D-3)\lambda\, r^{D-1}\,[2D^2-5D+6-3D(D-2)r^{D-3}] \\ &\quad +\tfrac{1}{4}(D+2)\,\lambda^2\, r^{2(D-1)}\,[3(D-2)-Dr^{D-1}] + \lambda^3\, r^{3(D-1)} \end{aligned}$$

$$\lambda \;=\; -m_2^2$$

Gregory-Laflamme Instability

Analysis of the possibility of growing $(\mathrm{Re}(\nu) > 0)$ perturbations can be done using WKB methods B.F. Schutz and C.M. Will, Ap.J. 291:L33 (1985) or numerically. But in fact, the answer has been known for some time from the 5D string R. Gregory and R. Laflamme, PRL 70 (1993) 2837 . Considering perturbations about the 5D black string $ds^2_{(5)} = ds^2_{(4)} + dz^2$

$$\begin{pmatrix} h^{(4)}_{\mu\nu} & h_{\mu z} \\ h_{z\nu} & h_{zz} \end{pmatrix} \tag{1}$$

where the z dependence is assumed to be of the form e^{ikz} one finds that $h^{(4)}_{\mu\nu}$ satisfies an equation of the same Lichnerowicz form equation $(\Delta_L + k^2)\, h^{(4)}_{\mu\nu} = 0$ as for $\delta R_{\mu\nu}$ Y.S. Myung, Phys.Rev. D88 (2013) . This form is also found for perturbations about the Schwarzschild solution in dRGT nonlinear massive gravity.

The Gregory-Laflamme instability is an S-wave ($\ell = 0$) spherically symmetric instability from the 4D perspective. In the higher-derivative theory, it exists for low-mass Schwarzschild black holes, which disappears for black hole masses $M \geq M_{\max}$ where

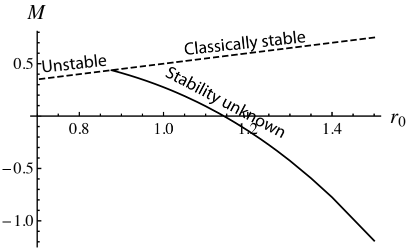
$$\frac{m_2 M_{\max}}{M_{\rm Pl}^2} = .438 \ .$$

This is precisely the branch point for the beginning of the family of new black holes.

Note that this monopole instability depends on the presence in the theory of the m_2 massive spin-two mode. In the $R + R^2$ theory, on the other hand, study of the quasinormal modes about the Schwarzschild solution shows it to be stable as long as the spin-zero mode mass is nontachyonic, $m_0^2 > 0$. This is perhaps not surprizing, since that theory is classically equivalent to ordinary Einstein gravity plus a scalar field with a peculiar potential, for which the ordinary GR stability considerations should apply.

Whitt, Starobinsky

We therefore have the following classical stability picture:



Classical stability regimes. The dashed line denotes Schwarzschild black holes and the solid line denotes non-Schwarzschild black holes.

Outlook

- Taking the fourth order field equations seriously for gravity including quadratic curvatures in the action leads to a rich space of asymptotically flat solutions including horizonless solutions, wormholes and both Schwarzschild and non-Schwarzschild black hole solutions.
- The branch of non-Schwarzschild black holes bears an intimate relation to black-hole stability: the branching point mass on the Schwarzschild family is also the upper limit of classical instability of the Schwarzschild solution.

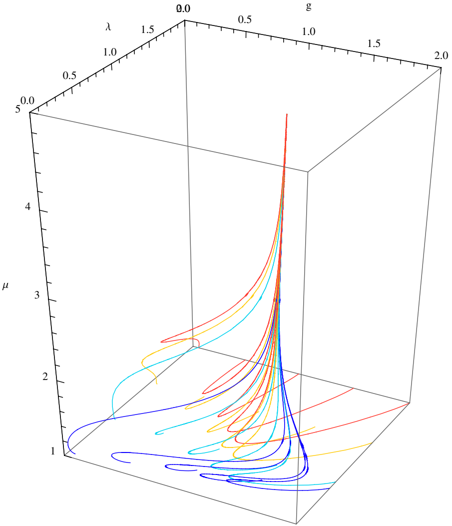
Important issues remaining:

- Stability of the non-Schwarzschild black holes themselves.
- Extension of this analysis to axisymmetric solutions.

Asymptotic Safety

A possible context for the occurrence of quadratic-curvature terms in the gravitational effective action is expressed in the proposal that gravity could be an asymptotically safe theory. Put forward initially by Steven Weinberg, this has given rise to a certain amount of discussion. M. Reuter 1996, M. Niedermaier 2009

The asymptotic-safety proposal extends the family of acceptable quantum theories beyond the strictly renormalizable ones to theories where there is a finite set of 'relevant' couplings lying on an ultraviolet critical surface within the (infinite) space of coupling constants. This includes ordinary renormalizable and asymptotically free theories, where there is a Gaussian fixed point at the origin of coupling-constant space, but can also include theories with non-trivial fixed points away from the origin, which would be of an essentially non-perturbative nature.



Renormalization-group trajectories in coupling-constant space ending on a non-Gaussian fixed point with finite g_{Newton} and cosmological constant Λ.

Niedermaier 2009

$R + R^2$ theory

The massive spin-two ghost can be eliminated at the classical level by setting $\alpha \to 0_+$, for which $m_2 \to \infty$. Choosing $\beta > 0$ makes the spin-zero mode non-tachyonic, and the resulting $\int d^4x\sqrt{-g}(-R + \beta R^2)$ theory is equivalent to GR coupled to a *non-ghost* scalar field K.S.S. 1978 . This remains true at the full nonlinear level B. Whitt 1984 , with an action (including also a cosmological term)

$$
\begin{aligned}
I_{R+\text{spin zero}} &= \int d^4x\sqrt{-g}(-R + \beta R^2 - 2\Lambda) \\
&\leftrightarrow \int d^4x\sqrt{-g}(-R \\
&\quad -6\beta^2(1+2\beta\phi)^{-2}(\nabla_\mu\phi\nabla^\mu\phi + \frac{1}{6\beta}\phi^2 + \frac{1}{3\beta^2}\Lambda))
\end{aligned}
$$

- One can redefine the scalar field $\phi = (e^{\tilde{\phi}/\sqrt{3}} - 1)/2\beta$ in order to produce a scalar Lagrangian with a canonical kinetic term and a transformed potential $-\frac{1}{2}\nabla_\mu\tilde{\phi}\nabla^\mu\tilde{\phi} - V(\tilde{\phi})$, where

$$V(\tilde{\phi}) = \frac{1}{4\beta}(1 - e^{-\tilde{\phi}/\sqrt{3}})^2 + 2\Lambda e^{-2\tilde{\phi}/\sqrt{3}}$$

- It is thus clear that, for large $\tilde{\phi}$, the potential $V(\tilde{\phi})$ becomes very flat. This was the reason for the attractiveness (at times) of the $\int d^4x\sqrt{-g}(-R + \beta R^2)$ theory for inflation purposes.

  A.A. Starobinsky 1980; Mukhanov & Chibisov 1981

  The coefficient β sets the scale for the potential. Restoring a $1/\kappa^2$ coefficient for the Einstein-Hilbert action $\int\sqrt{-g}R$, the mass of the scalar mode is $m_0^2 = (6\kappa^2\beta)^{-1}$; applications for inflation typically take this mass scale to be something like 10^{-6} of the Planck scale.