Higher spins and topological open strings

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June 2016

First-quantized higher spin gravity

The key formula

$$\langle J_{\chi_1}(x_1)\cdots J_{\chi_n}(x_n)\rangle_{\text{free CFT3}} = \sum_{\text{permutations}} \langle V_{x_1,\chi_1}(\tau_1)\cdots V_{x_n,\chi_n}(\tau_n)\rangle_{\text{TFT2}} \ ,$$

LHS: correlation functions of bi-linear operators

$$J_\chi(x) = \sum_{s\geqslant 0} J^{(s)}_{\mu_1\dots\mu_s}(x)\bar{\chi}\sigma^{\mu_1}\chi\cdots\bar{\chi}\sigma^{\mu_s}\chi \ ,$$

computed in three-dimensional free conformal field theory on $\mathbb{R}^{1,2}$, labelled here by vectorial coordinates x and $Sp(2,\mathbb{R})$ spinors χ.

RHS: correlation functions of Gaussian vertex operators

$$V_{x,\chi}(\tau) = \exp(4iy(\tau)\sigma^a\bar{y}(\tau)) \ ,$$

computed in three-dimensional free conformal field theory on $\mathbb{R}^{1,2}$,

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2 High energy strings in AdS

The singleton alias Dirac's conformal particle The simplest nontrivial unitary representation of $\mathfrak{iso}(2,D-1)$ with $D \geqslant 4$, is the lowest weight space

$$D(\tfrac{1}{2}(D-3);0) = \bigoplus_{n\geqslant 0} L^+_{r_1}\cdots L^+_{r_n} \mid \tfrac{1}{2}(D-3);0\rangle \ ,$$

referred to as the singleton, generated by acting with ladder operators $L^\pm_r$ ($r = 1,\dots,D-1$) on a spinless ground state with energy $\frac{1}{2}(D-3)$, *viz.*

$$L^-_r \mid \tfrac{1}{2}(D-3);0\rangle = 0 \ , \qquad (E-\tfrac{1}{2}(D-3)) \mid \tfrac{1}{2}(D-3);0\rangle = 0 \ , \qquad M_{rs} \mid \tfrac{1}{2}(D-3);0\rangle = 0 \ ,$$

and factoring out the singular vector

$$\delta^{rs} L^+_r L^+_s \mid \tfrac{1}{2}(D-3);0\rangle \approx 0 \ .$$

The singleton can be realized as a conformal particle in $D-1$ dimensions, that is, as a particle with $2(D-2)$-dimensional phase space obtained by factoring out an $\mathfrak{sp}(2)$-algebra from a $2(D+1)$-dimensional phase space, *viz.*

$$[X^A, P_B] = i\hbar\delta^A_B \ , \qquad M_{AB} = X_A P_B - X_B P_A \ ,$$

$$X^A X^B \eta_{AB} \approx 0 \ , \qquad X^A P_A - \tfrac{i\hbar}{2}(D+1) \approx 0 \ , \qquad P_A P_B \eta^{AB} \approx 0 \ .$$

The constraint surface, which is known as Dirac's hypercone, is thus the limit of D-dimensional anti-de Sitter spacetime in which the anti-de Sitter radius L vanishes. The singletons is sometimes said to be hyper-light-like in order to distinguish it from the light-like (massless) particle, whose $2(D-1)$-dimensional phase space is obtained by factoring out

$$X^A X^B \eta_{AB} \approx -L^2 \ , \qquad X^A P_A - \tfrac{i\hbar}{2}(D+1) \approx 0 \ , \qquad P_A P_B \eta^{AB} \approx 0 \ .$$

The space of composite operators on the singleton phase space is isomorphic to the space $\mathcal{A}$ of functions in $\mathrm{Env}(\mathfrak{iso}(2, D-1))$ over the two-sided ideal generated by

$$\tfrac{1}{2} M_{AC} M_B^C + \tfrac{1}{4}(D-3)\eta_{AB} \approx 0 \ , \qquad M_{[AB} M_{CD]} \approx 0 \ .$$

The singleton arises inside this associative algebra as two one-sided modules generated from a special function from the left or the right; for example, in $D = 4$ the singleton ground state arises as the following coherent state density matrix:

$$| \tfrac{1}{2}(D-3); 0 \rangle\langle \tfrac{1}{2}(D-3); 0 | = 4 \exp(-4E) \ .$$

Flato-Fronsdal theorem The massless simple spin-s ($s \geqslant 1$) particle in D-dimensional anti-de Sitter spacetime is the lowest weight space

$$D(s+D-3;(s)) = \bigoplus_{n\geqslant 0} L^+_{r_1}\cdots L^+_{r_n} \mid s+D-3;(s)\rangle_{s_1\dots s_s} \ ,$$

generated from a spin-s ground state with energy $s+D-3$ modulo the singular vector

$$\delta^{rs} L^+_r \mid s+D-3;(s)\rangle_{ss_2\dots s_s} \ .$$

Remarkably, as first found by Flato and Fronsdal in $D=4$, these massless particles are composite in the sense that they are contained in the direct product of two scalar singletons, *viz.*

$$D(\tfrac{1}{2}(D-3);0)\otimes D(\tfrac{1}{2}(D-3);0) = \bigoplus_{s=0}^{\infty} D(s+D-3;(s)) \ ,$$

where all irreps on the right-hand side have multiplicity one and the even spin arises in the symmetric product and the odd spin in the anti-symmetric product. Thus, massless particles arise as nonpolynomial elements V in $\mathcal{A}$, that is, as singleton density matrices, forming $\mathfrak{iso}(2,D-1)$-orbits under the twisted adjoint action of $\mathcal{A}$

on itself induced via

$$\rho(M_{AB})V = M_{AB}V - V\pi(M_{AB}) \ , \qquad \pi(E, M_{rs}, L_r^{\pm}) = (-E, M_{rs}, L_r^{\mp}) \ .$$

As we shall see, the above realization is the germ of stringy holography, as conformal particles can be realized as cusps on closed strings in anti-de Sitter spacetime, as well as of the topological open string as, singleton density matrices can be realized as boundary vertex operators of first-quantized two-dimensional topological sigma models.

Semiclassical singletons as cusps on tensionful strings The $\mathfrak{iso}(2, D-1)$ Noether current densities of semiclassical tensionful closed strings in D-dimensional anti-de Sitter spacetime have a tendency to accumulated in the semi-classical limit (large charges) at relatively small portions of the string moving just below the light-speed. Assuming the (principal) spin $S \gg TL^2 \gg 1$, the simplest string configuration is a folded string with energy

$$E = S + \mathcal{O}(\log S) \ ,$$

and isometry charges distributed equally among two portions of fixed size much smaller than the size of the string. On physical grounds, we may thus think of a generic semi-classical string state as consisting of N cyclically ordered hyper-light-like cusps with $\mathfrak{iso}(2, D-1)$ charges $M_{AB}(\xi)$ $(\xi = 1, \ldots, N)$ such that

$$M_{AB} = \sum_{\xi=1}^{N} M_{AB}(\xi) \ , \qquad M_{AC}(\xi) M_B{}^C(\xi) \sim \mathcal{O}(\log S_\xi) \ .$$

The cusps receive one-loop corrections from $D-2$ normal-coordinate bound states trapped in Poeschl-Teller Type II potential wells; $D-3$ of these have zero-point energy $\frac{1}{2}$ while the energy of the remaining one depends on the details of the geometry of the cusp.

Singletons as partons of discretized tensionless strings on Dirac hypercone

The Nambu-Goto action for the a bosonic string with tension T in D-dimensional anti-de Sitter spacetime can be written as

$$S = T_s \int d^2\sigma (\sqrt{-\gamma} \gamma^{\alpha\beta} \partial_\alpha X^A \partial_\beta X^B \eta_{AB} + \Lambda (X^A X^B \eta_{AB} + L^2)) \ ,$$

where $\gamma_{\alpha\beta}$ is the Polyakov metric and Λ is a Lagrange multiplier. In the Polyakov gauge, the corresponding Hamiltonian action with rescaled spatial coordinate reads

$$S = \int d\tau \oint d\mu (P_A \dot{X}^A - \Lambda_1 (X^A X^B \eta_{AB} + L^2) - \Lambda_2 P \cdot \partial_\mu X - \Lambda_3 (P^2 + (T_s)^2 (\partial_\mu X)^2)) \ .$$

Replacing the closed string by N partons with coordinates

$$Y_I^A = (X^A(\mu_\xi), P^A(\mu_\xi)) \ , \qquad \mu_\xi = \xi/a \ , \qquad I = 1, \ldots, 2N \ ,$$

where $1/a$ is the unit string mass per parton, and taking the combined tensionless and hyperconical limit

$$L/a \to 0 \ , \qquad Ta^2 \to 0 \ ,$$

at fixed a, while requiring a constraint algebra containing the original constraints, yields an $\mathfrak{sp}(2N)$-gauged quantum mechanical system with action

$$S = \int d\tau (\tfrac{1}{2} \Omega_{IJ} Y^{AI} \dot{Y}_A^J - \Lambda_{IJ} K^{IJ}) \ , \qquad K^{IJ} = Y^{AI} Y_A^J \ ,$$

where one may think of $\mathfrak{sp}(2N)$ as a discretized W-algebra.

Although the classical phase space becomes overconstrained (for $N > 1$), requiring that physical states obey the weaker constraint

$$\langle \Psi \mid K^{IJ} \mid \Psi \rangle = 0 \ ,$$

leads to a reincarnation of multi-singleton states

$$\mid \Psi \rangle = \bigotimes_{\xi=1}^{N} \mid \Psi_\xi \rangle_\xi \ , \qquad K^{ij}(\mu_\xi) \mid \Psi_\xi \rangle_\xi = 0 \ , \qquad K^{ij} = \left[\begin{array}{cc} X^2 & \frac{1}{2}(XP + PX) \\ \frac{1}{2}(XP + PX) & P^2 \end{array} \right] ,$$

though additional ghosts are required in order to cancel the $\mathfrak{sp}(2N)$-anomaly, *i.e.* to make

$$\langle \Psi \mid K^{I_1 J_1} \cdots K^{I_n J_n} \mid \Psi \rangle$$

vanish.

A mechanism for the Hagedorn transition? As $Sp(2N)$ contains S_N, the invariance under large gauge transformations implies a spectrum of totally symmetric multi-singleton states, that is, a tensionless bosonic string in D-dimensional anti-de Sitter spacetime appears to be dual to a single conformal scalar field on the boundary of spacetime.

Thus, returning to the tensionful string, it appears as if the cusps can actually pass trough each other in the limit of large quantum numbers. This is reminiscent of what one would expect from nonlinear solitons (the Poescl–Teller Type II problem is indeed the auxiliary system for the KdV equation).

We may thus interpret the formation of cusps as a mechanism for a Hagedorn transition: Start from closed string states on Regge trajectories with low energy (in flat spacetime); switch on a negative cosmological constant small enough not to affect the system significantly; heat up the system thereby exciting cyclically ordered string states on Regge trajectories; at high enough temperature, the energy starts going into totally symmetric cusps (feeling the curvature), and the string eventually becomes tensionless in anti-de Sitter spacetime.

Two-parton system and $sp(2)$-gauged topological open string The two-parton system can be modelled as a single closed worldline with action

$$S = \oint d\tau (\tfrac{1}{2}\epsilon_{ij} Y^{Ai} \dot{Y}^j_A - \Lambda_{ij} K^{ij}) \ , \qquad K^{ij} = Y^{Ai} Y^j_A \ , \qquad Y^A_i = (X^A, P^A) \ ,$$

which can be rewritten as a two-dimensional gauged Poisson sigma model with covariant Hamiltonian action

$$S = \int_\Sigma (\eta_{Ai} \wedge (DY^{Ai} + \tfrac{1}{2}\eta^{Ai}) + \xi_{ij} F^{ij} + B^{ij}(\xi_{ij} + K_{ij}),$$

$$DY^{Ai} = dY^{Ai} + \Lambda^{ij} Y_j^A \ , \qquad F^{ij} = d\Lambda^{ij} + \Lambda^{ik} \wedge \Lambda_k{}^j,$$

whose AKSZ treatment requires the boundary conditions

$$\eta_{Ai}|_{\partial\Sigma} = 0 \ , \qquad \xi_{ij}|_{\partial\Sigma} = 0 \ .$$

The emission and absorbing massless states $|i\rangle D(\frac{1}{2}(D-3);0) \otimes D(\frac{1}{2}(D-3);0)$ can be modelled by inserting corresponding adjointized vertex operators $V_i \star \kappa$ where V_i is a twisted adjoint singleton density matrix and κ is the inner Klein operator implementing the π-map, *viz.*

$$\kappa V = \pi(V)\kappa \ .$$

The resulting, naive, form of massless particle scattering amplitudes is given by traces of the form

$$\mathcal{C}_{1\ldots n} = \sum_{\text{bose}-\text{symm}} \text{Tr} \prod_{i=1}^{n} V_i \star \kappa \ ,$$

which indeed reproduce free field theory correlation functions.

3 Topological open strings I

Ikeda-Schaller-Strobl sigma model The sigma model map

$$\varphi : M \equiv T[1]\Sigma \to N \equiv T^*[1]P \ ,$$

the parity shifted cotangent bundle over a Poisson manifold P with bi-vector Π obeying $[\Pi, \Pi]_{NS} = 0$. The cotangent bundle is a graded symplectic manifold with tautological one-form

$$\vartheta = \eta_\alpha \wedge d\phi^\alpha \ , \qquad \deg(\phi^\alpha, \eta_\alpha, d, \vartheta) = (0, 1, 1, 2) \ ,$$

canonical two-form

$$\Omega = d\eta_\alpha \wedge d\phi^\alpha \ , \qquad \deg(\Omega) = 3 \ ,$$

and corresponding Poisson bracket $\{\cdot, \cdot\}_{[1]}$ of intrinsic degree 1. The Poisson bi-vector field on P lifts to a Hamitonian function on $T^*[1]P$ of degree 2:

$$H := \frac{1}{2}\eta_\alpha \eta_\beta \Pi^{\alpha\beta}(\phi) \ , \qquad \{H, H\}_{[1]} = 0 \ , \qquad \deg(H) = 2 \ .$$

The sigma model action

$$S = \int_M \varphi^*(\vartheta + H) \ ,$$

where $\int_M$ involves a projection onto horizontal forms and assuming that the sigma model map, which is a map between graded manifolds, has vanishing intrinsic degree and that the degree map on M corresponds to the form degree on Σ; more explicitly, and suppressing φ^*, we have

$$S = \int_\Sigma (\eta_\alpha \wedge d\phi^\alpha + \frac{1}{2}\eta_\alpha \wedge \eta_\beta \Pi^{\alpha\beta}(\phi)) \ .$$

AKSZ-Cataneo-Felder quantization and Kontsevich's formality theorem The above model is a special example of a class of formally topological models based on generally covariant actions in $p+1$ dimensions of the form

$$S = \int_M (P_\alpha \wedge dX^\alpha + H(P,X)) \ , \qquad \{H,H\}_{[p]} = 0 \ ,$$
$$\deg(X^\alpha, P_\alpha, H) = (p_\alpha, p - p_\alpha, p+1) \ ,$$

that can be quantized using a special implementation of the Batalin-Vilkovisky formalism due to Alexandrov-Kontsevich-Schwartz-Zaboronsky. Following this method, Cataneo and Felder quantized the Ikeda-Schaller-Strobl model in the case that P has a trivial topology ($\mathbb{R}^n$) and Σ is a disc with

$$\eta_\alpha|_{\partial\Sigma} = 0 \ ,$$

leading to an explicit verification of Kontsevich's associative star product formula; if ϕ_i are functions on P then

$$(\phi_1 \star \phi_2)(p) = \int_{\substack{\text{gaugeslice} \\ \varphi(q_0)=p \\ \eta|_{\partial\Sigma}=0}} D(\varphi, \cdots) e^{iS_{\text{AKSZ}}/\hbar} \phi_1(\varphi(q_1))\phi_2(\varphi(q_2)) ,$$

where $p \in P$ and q_0, q_1 and q_2 are three cyclically ordered points at $\partial\Sigma$, the path integral involves ghosts and Lagrange multipliers and S_{AKSZ} is the BV master action obtained using the AKSZ method. Perturbative quantization using the background field method and and choosing a regularization scheme involving deletion of self-contracted vertices (normal ordering) reproduces Konsevich's explicit formula

$$\phi_1 \star \phi_2 = \phi_1\phi_2 + \tfrac{i\hbar}{2}\{\phi_1, \phi_2\}_\Pi + \sum_{n\geqslant 2} \tfrac{(i\hbar)^n}{n!} \sum_{\Gamma\in\Gamma_{n,2}} w_\Gamma B_\Gamma(\phi_1, \phi_2) ,$$

where $\Gamma_{n,2}$ are all Kontsevich's diagrams with n vertices and w_Γ are the celebrated Kontsevich weights.

Remarks: Formal quantization vs quantum states and vertex operators? Kontsevich's formula gives a star product that is associative if all objects, including

ϕ_i, admit perturbative $\hbar$-expansions. Thus, the formula provides a definition of a quantum algebra but not necessarily a practical tool for handling physically relevant representations, as these generically involve functions that depend nonperturbatively on $\hbar$. Moreover, Kontsevich's explicit formula is not manifestly coordinate invariant.

Zucchini's gauged model Let V_r be fundamental vector fields on P, of which an (ideal) subalgebra generated by $V_{r'} = P^r_{r'} V_r$ admits moment functions $K_{r'}$ (Hamiltonians). Zucchini proposed the following gauged version of the Poisson sigma model:

$$S = \int_\Sigma (\eta_\alpha \wedge (d\phi^\alpha + \lambda^r V_r^\alpha) + \frac{1}{2}\eta_\alpha \wedge \eta_\beta \Pi^{\alpha\beta} + \xi_r (d\lambda^r f^r_{st} \lambda^s \wedge \lambda^t + B^{r'}(P^r_{r'}\xi_r + \mu_{r'}) ,$$

together with the boundary conditions

$$(\eta_\alpha, \xi_r)|_{\partial\Sigma} = 0 .$$

Formally, the corresponding quantum structure is that of a star product compatible with a nilpotent BRST differential, *viz.*

$$Q(\Phi_1 \star \Phi_2) = (Q\Phi_1) \star \Phi_2 + (-1)^{\mathrm{gh}(\Phi_1)} \Phi_1 \star (Q\Phi_2) , \qquad Q(Q(\Phi)) = 0 ,$$

subject to potential anomalies provoked by the Cattaneo-Felder subtraction scheme.

4 Higher spin gravity

Twisted-adjoint representation and Weyl zero-form

Zero-form charges and amplitudes

Adjoint representation and one-form connection

Perturbative unfolded approach vs Vasiliev equations

Emergence of deformed Poisson structures

Dynamical two-form and Frobenius-Chern-Simons model

Remark: Free energy proposal

5 Topological open strings II

Differential Poisson manifolds A differential Poisson algebra on a manifold P is an extension of a Poisson algebra on P to the algebra $\Omega(P)$ of differential forms on P that admits two degree-preserving bilinear products, given by the graded commutative wedge product and the graded skew-symmetric map

$$\{\cdot,\cdot\} : \Omega(P)\otimes\Omega(P) \longrightarrow \Omega(P) \ ,$$

referred to as Poisson bracket, which is assumed to i) be compatible with the de Rham differential; ii) obeys the graded Leibniz rule, *viz.*

$$\begin{aligned}
\deg_P(\{\omega_1,\omega_2\}) &= \deg_P(\omega_1)+\deg_P(\omega_2) \ ,\\
\{\omega_1,\omega_2\} &= (-1)^{1+\deg_P(\omega_1)\deg_P(\omega_2)}\{\omega_2,\omega_1\} \ ,\\
\{\omega_1,\omega_2+\omega_3\} &= \{\omega_1,\omega_2\}+\{\omega_1,\omega_3\} \ ,\\
d\{\omega_1,\omega_2\} &= \{d\omega_1,\omega_2\}+(-1)^{\deg_P(\omega_1)}\{\omega_1,d\omega_2\} \ ,\\
\{\omega_1,\omega_2\wedge\omega_3\} &= \{\omega_1,\omega_2\}\wedge\omega_3+(-1)^{\deg_P(\omega_1)\deg_P(\omega_2)}\omega_2\wedge\{\omega_1,\omega_3\} \ ;
\end{aligned}$$

and iii) obeys the graded Jacobi identity

$$
\begin{aligned}
\{\omega_1,\{\omega_2,\omega_3\}\} \quad + \quad & (-1)^{\deg_P(\omega_1)(\deg_P(\omega_2)+\deg_P(\omega_3))}\{\omega_2,\{\omega_3,\omega_1\}\} \\
& +(-1)^{\deg_P(\omega_3)(\deg_P(\omega_1)+\deg_P(\omega_2))}\{\omega_3,\{\omega_1,\omega_2\}\} = 0 \ ,
\end{aligned}
$$

where $\omega_i \in \Omega(P)$ and $\deg_P$ denotes the form degree on $\Omega(P)$. The differential Poisson bracket can be covariantized by introducing a connection one-form $\widetilde{\Gamma}^\alpha{}_\beta = d\phi^\gamma \, \Gamma^\alpha_{\gamma\beta}$ and a tensorial one-form

$$S = \tfrac{1}{2}\, d\phi^\alpha \, S^{\beta\gamma}_\alpha \partial_\beta \odot \partial_\gamma \ .$$

defined through

$$\{\phi^\alpha, d\phi^\beta\} = \tfrac{1}{2}\widetilde{\nabla}\Pi^{\alpha\beta} + S^{\alpha\beta} - \Pi^{\alpha\gamma}\widetilde{\Gamma}^\beta{}_\gamma \ .$$

Choosing the connection such that[1]

$$\widetilde{\nabla}_\gamma \Pi^{\alpha\beta} = 0$$

[1]There exists a generalization of Darboux's theorem from symplectic to Poisson manifolds that makes this choice possible.

and imposing the de Rham-compatibility, it follows that, for any two differential forms ω and η, one has

$$\{\omega,\eta\} = \Pi^{\alpha\beta}\nabla_\alpha\omega\wedge\nabla_\beta\eta + S^{\alpha\beta}\wedge\left[(-1)^{\deg_P(\omega)}\nabla_\alpha\omega\wedge i_\beta\eta - i_\alpha\omega\wedge\nabla_\beta\eta\right]$$
$$+ (-1)^{\deg_P(\omega)}\left(\widetilde{R}^{\alpha\beta} - \tilde{\nabla}S^{\alpha\beta}\right)\wedge i_\alpha\omega\wedge i_\beta\eta \ ,$$

where ∇_α is the covariant derivative constructed from the connection coefficients

$$\Gamma^\alpha_{\gamma\beta} = \widetilde{\Gamma}^\alpha_{\beta\gamma} \ ,$$

and

$$\widetilde{R}^{\alpha\beta} := \Pi^{\beta\gamma}\widetilde{R}^\alpha{}_\gamma = \widetilde{R}^{\beta\alpha} \ ,$$

where $\widetilde{R}^\alpha{}_\beta$ is the two-form curvature of the connection one-form $\widetilde{\Gamma}^\alpha{}_\beta$.

If $S = 0$, the graded Jacobi identity (5.1) is equivalent to following conditions:

$$\begin{aligned}
J_0^{\alpha\beta\gamma} &:= \Pi^{\delta[\alpha}T^\beta_{\delta\epsilon}\Pi^{\gamma]\epsilon} = 0 \ , \\
J_1^{\alpha\beta,\gamma}{}_\delta &:= \Pi^{\alpha\rho}\Pi^{\star\beta}R_{\rho\star}{}^\gamma{}_\delta = 0 \ , \\
J_2^{\alpha,\beta\gamma}{}_{\delta\epsilon} &:= \Pi^{\alpha\lambda}\nabla_\lambda\widetilde{R}_{\beta\gamma}{}^{\rho\star} = 0 \ , \\
J_3^{\alpha\beta\gamma}{}_{\delta\epsilon\lambda} &:= \widetilde{R}_{\epsilon[\rho}{}^{(\alpha\beta}\widetilde{R}_{\star\lambda]}{}^{\gamma)\epsilon} = 0 \ .
\end{aligned}$$

of which the constraints on $J_0^{\alpha\beta\gamma}$, $J_1^{\alpha(\beta,\gamma)}{}_\delta$ and $J_2^{[\alpha,\beta]\gamma}{}_{\delta\epsilon}$ are independent, whereas the remainder follows by covariant differentiation.

Induced supersymmetric differential Poisson sigma model The isomorphism between the algebra of forms on P and functions on $T[1]P$, both of which are graded associative differential algebras, can be exploited to construct a supersymmetric extension of the Ikeda–Schaller–Strobl sigma model by fermionic $(0,1)$-forms $(\theta^\alpha, \chi_\alpha)$ in the case that P is a differential Poisson manifold with $S = 0$.

The action, which thus exhibits a global supersymmetry corresponding to the de Rham differential on P, and whose gauge symmetries are equivalent to the constraints required for the Jacobi identity, is given by

$$S[\phi^\alpha, \eta_\alpha, \theta^\alpha, \chi_\alpha] = \int_\Sigma \varphi^* \left(\eta_\alpha d\phi^\alpha + \tfrac{1}{2}\Pi^{\alpha\beta}\eta_\alpha\eta_\beta + \chi_\alpha \nabla\theta^\alpha + \tfrac{1}{4}\widetilde{R}_{\gamma\delta}{}^{\alpha\beta}\chi_\alpha \wedge \chi_\beta\theta^\gamma\theta^\delta \right) ,$$

where $\varphi : M = T[1]\Sigma \to N = T^*[1]P$ is a degree preserving sigma model map and the $(\phi^\alpha, \eta_\alpha; \theta^\alpha, \chi_\alpha)$ are assigned a target space degrees, denoted by deg, and fermion numbers as follows

	ϕ^α	η_α	θ^α	χ_α	d
deg	0	1	0	1	1
$\epsilon_{\rm f}$	0	0	1	-1	0

The target space thus consists of a bi-graded fiber bundle

$$T^*[1,0]P \oplus T^*[1,1]P \oplus T[0,1]P \ ,$$

with base manifold P coordinatized by ϕ^α and fiber coordinatized by $(\eta_\alpha, \chi_\alpha, \theta^\alpha)$, respectively.

The pull-back by φ^* induces the following bundle structure over Σ:

$$\varphi^*\Big(T^*[1,0]P \oplus T^*[1,1]P \oplus T[0,1]P\Big) = \Big(\varphi^*(T^*[0,0]P) \otimes T^*\Sigma\Big) \\ \oplus \Big(\varphi^*(T^*[0,1]P) \otimes T^*\Sigma\Big) \oplus \Big(\varphi^*(T[0,1]P) \otimes C^\infty(\Sigma)\Big) \ ,$$

whose sections we shall denote simply by $(\eta_\alpha, \chi_\alpha, \theta^\alpha)$, respectively.

The suitable Koszul sign convention reads[2]

$$\mathcal{O}\mathcal{O}' = (-1)^{(\deg(\mathcal{O})+\epsilon_{\rm f}(\mathcal{O}))(\deg(\mathcal{O}')+\epsilon_{\rm f}(\mathcal{O}'))}\mathcal{O}'\mathcal{O} \ .$$

The equations of motion of the supersymmetric Poisson sigma model, *viz.*

$$\mathcal{R}^{\phi^\alpha} := d\phi^\alpha + \Pi^{\alpha\beta}\eta_\beta \approx 0 \ , \tag{5.1}$$

$$\mathcal{R}^{\theta^\alpha} := \nabla\theta^\alpha + \tfrac{1}{2}\widetilde{R}_{\gamma\delta}{}^{\alpha\beta}\chi_\beta\theta^\gamma\theta^\delta \approx 0 \ , \tag{5.2}$$

$$\mathcal{R}^{\eta_\alpha} := \nabla\eta_\alpha + R_{\alpha\beta}{}^\gamma{}_\delta\, d\phi^\beta \wedge \chi_\gamma\theta^\delta + \tfrac{1}{4}\nabla_\alpha\widetilde{R}_{\beta\gamma}{}^{\delta\epsilon}\chi_\delta \wedge \chi_\epsilon\theta^\beta\theta^\gamma \approx 0 \ , \tag{5.3}$$

$$\mathcal{R}^{\chi_\alpha} := \nabla\chi_\alpha + \tfrac{1}{2}\widetilde{R}_{\alpha\delta}{}^{\beta\gamma}\chi_\beta \wedge \chi_\gamma\theta^\delta \approx 0 \ , \tag{5.4}$$

form a universally Cartan integrable system, and the action is gauge invariant, by virtue of the constraints following from the requirement that the differential Poisson bracket obeys the graded Jacobi identity.

[2] This convention is motivated by the fact that it admits a direct extension to generalized Poisson sigma models in higher dimensions with target spaces given by $\mathbb{N}$-graded manifolds.

The (rigid) supersymmetry transformations

$$\begin{aligned} \delta_{\mathrm{f}}\phi^\alpha &= \theta^\alpha \ , \\ \delta_{\mathrm{f}}\theta^\theta &= 0 \ , \\ \delta_{\mathrm{f}}\eta_\alpha &= \tfrac{1}{2}\widetilde{R}_{\beta\gamma}{}^{\delta}{}_{\alpha}\,\chi_\delta\,\theta^\beta\theta^\gamma - \Gamma^\gamma_{\alpha\beta}\,\eta_\gamma\,\theta^\beta \ , \\ \delta_{\mathrm{f}}\chi_\alpha &= -\eta_\alpha + \Gamma^\gamma_{\alpha\beta}\,\chi_\gamma\,\theta^\beta \ , \end{aligned} \tag{5.5}$$

are nilpotent and leave the action (5) invariant, as can be seen by writing

$$S = \int_\Sigma \delta_{\mathrm{f}} V \ , \quad V = -\chi_\alpha \wedge \left(d\phi^\alpha + \tfrac{1}{2}\Pi^{\alpha\beta}\eta_\beta\right) \ .$$

The overall coefficient of the term in the action that is quartic in fermions is fixed by the supersymmetry but not the local symmetries. The corresponding Noether current is

$$J_{\mathrm{f}} = \eta_\alpha\theta^\alpha - \tfrac{1}{2}T^\alpha_{\beta\gamma}\chi_\alpha\theta^\beta\theta^\gamma \ .$$

Gauging the target space de Rham differential In order to construct a topological open string theory describing higher spin gravity, we propose to gauge the de Rham differential, that is, the rigid vectorial supersymmetry of the differential Poisson sigma model.

At the semi-classical level, the consistency of the construction does not impose any further constraints on the differential Poisson geometry than those required for the ungauged model.

The resulting action

$$S_\psi = S_0 + \int_\Sigma (\psi \wedge J_{\rm f} + \lambda d\psi + c\psi \wedge \psi) \ .$$

Towards nonlinear quantum mechanics (work in progress) We are currently investigating a number of basic properties of the model:

- Perturbative quantization of model by rewriting the action as a sigma model on a supermanifold with a Poisson bi-supervector.
- Derivation of manifestly covariant extension of Kontsevich star product formula on to differential forms.

- Presence of anomalies in $(Q_{\rm f})^2$.
- Anomalies in more general (supersymmetric) gaugings (*e.g.* for extra supersymmetry induced by special Killing vectors).
- Application differential Poisson manifolds with nontrivial curvature tensor provided by (semi-classical) quantum groups G_q.

In particular, we would like to understand better whether it is possible to embed the ordinary Liouville equation

$$i\hbar\dot{\rho} + [H,\rho]_\star = 0 \ ,$$

into a linearized open string field theory equation of motion

$$Q\Psi = 0 \ ,$$

which may pave the way towards an example of a nonlinear deformation of quantum mechanics.

6 Conclusion

- CFT correlation functions from FCS free energy functional?
- CFT correlation functions from TOS amplitudes?
- TOS $\rightarrow$ TOSFT $\rightarrow$ FCS ?
- Liouville equation as linearized gauged TOSFT ?