Consistent truncations and the duality hierarchy

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Dualities in supergravitites, strings and branes

Texas A&M, 15 April 2016

Based on

- A. Guarino, D. Jafferis, OV, Phys. Rev. Lett. **115** (2015) 9 [arXiv:1504.08009]
- A. Guarino, OV, arXiv:1508.04432
- A. Guarino, OV, arXiv:1509.02526
- OV, arXiv:1509.07117
- OV, arXiv:1512.04943

Three puzzles

Puzzle 1: AdS/CFT for the simplest CS?

- A natural way to obtain a CFT_3 is to start from Chern-Simons (CS) theory and add couplings. CS with a simple gauge group $\text{SU}(N)$ and adjoint matter seemed like a good candidate to describe the M2-brane CFT_3.

  However: [Schwarz '04]

- Such CFT_3s cannot preserve maximal supersymmetry.

- The CS term seems to be related to the Romans mass.

Puzzle 1: AdS/CFT for the simplest CS?

- The M2-brane CFT_3 is now known to be described by an ($\mathcal{N} = 6$) CS-matter theory with non-simple gauge group $\mathrm{SU}(N) \times \mathrm{SU}(N)$ at levels k and $-k$.

  [Aharony, Bergman, Jafferis, Maldacena '08]

- The question remained: do the simplest type of CS-matter theory enjoy AdS_4 duals? Prospects looked bleak. For most of these theories the spectrum has light higher spin operators and exponential growth. [Minwalla, Narayan, Sharma, Umesh, Yin '11]

- Some of these simplest CS-matter theories can still have conventional AdS_4 duals. But none has been found until now.

Puzzle 2: AdS_4 backgrounds of type IIA

- M-theory and type IIB have many known AdS_4 and AdS_5 backgrounds, respectively, in and beyond the Freund-Rubin class.

- This is related to the presence of $\hat{F}_{(4)}$ and $\hat{F}_{(5)}$ in the respective field contents. This is in turn related to the fact that the dual CFTs should be conformal phases of the M2 and D3 brane field theories.

- Massless and massive type IIA also have an $\hat{F}_{(4)}$. However, excluding the massless IIA solutions obtained from M-theory on S^1, essentially only one class of AdS_4 solutions, of massive IIA, is explicitly know analytically.

Puzzle 2: AdS_4 backgrounds of type IIA

- This is the class of direct products $\mathrm{AdS}_4 \times M_6$, where M_6 is nearly-Kähler, and where the IIA forms take values along the nearly-Kähler forms on M_6. [Behrndt, Cvetic '04] This class generalises to M_6 half-flat. [Lüst, Tsimpis '04]

- These manifolds can be thought to be six-dimensional counterparts of five-dimensional Sasaki-Einstein manifolds.

- However, while infinitely many Sasaki-Einstein five-manifolds are known, *e.g.* in the cohomogeneity-one class, [Gauntlett, Martelli, Sparks, Waldram '04] the only explicitly known nearly-Kähler manifolds are homogeneous. Similarly, only homogeneous examples are known in the half-flat case [Koerber, Lüst, Tsimpis '08]

- Generalisations with $\mathrm{SU}(3) \times \mathrm{SU}(3)$ structure can be studied [Lüst, Tsimpis '09] but there is no known analytical example in massive IIA (see however [Rota, Tomasiello '15])

Puzzle 3: dyonic $\mathcal{N}=8$ supergravity

- $D=4$ $\mathcal{N}=8$ gauged supergravity often admits continuous or discrete symplectic deformations that respect $\mathcal{N}=8$ supersymmetry and the gauge group

  [Dall'Agata, Inverso, Trigiante, '12]

- E.g. the covariant derivatives acquire a new coupling to the magnetic vectors proportional to a parameter c,

$$D = d - g\,(\mathcal{A}^{\Lambda} - c\,\tilde{\mathcal{A}}_{\Lambda})\ .$$

- At finite gauge coupling g, electric/magnetic duality is broken and the theory typically becomes sensitive to the symplectic frame specified by c. The physical couplings of the supergravity develop a c dependence.

Puzzle 3: dyonic $\mathcal{N} = 8$ supergravity

- Do these $\mathcal{N} = 8$ gaugings enjoy a string or M-theory origin? For dyonic gaugings with AdS vacua, do these have CFT_3 duals?

- For example, the purely electric $\mathcal{N} = 8$ SO(8) gauging arises from consistent truncation of $D = 11$ supergravity on S^7. [de Wit, Nicolai '87]

- All the solutions of the $D = 4$ theory give rise to solutions in $D = 11$. In particular, the $D = 4$ vacua uplift to $\mathrm{AdS}_4 \times S^7$ M-theory backgrounds.

- Some of these have known CFT_3 duals. Eg, the central vacuum uplifts to Freund-Rubin, which is dual to ABJM.

Our new results

- Massive type IIA supergravity admits an $\mathcal{N}=8$ consistent truncation on S^6.

- The resulting $D=4$ theory has a dyonically-gauged $\mathrm{ISO}(7)=\mathrm{SO}(7)\ltimes\mathbb{R}^7$ gauge group.

- All the solutions of the $D=4$ theory give rise to solutions in $D=10$. In particular, the $D=4$ vacua uplift to $\mathrm{AdS}_4\times S^6$ massive type IIA backgrounds.

  We found the first explicit $\mathcal{N}=2$ such solution, a new $\mathcal{N}=1$ solution and recover other solutions. All of these have the $\mathrm{SU}(3)\times\mathrm{SU}(3)$ structure of [Lüst, Tsimpis '09].

Our new results

- Massive type IIA on these $\mathrm{AdS}_4 \times S^6$ backgrounds is dual to the simple CS theories of type discussed above. We gave the first $\mathrm{AdS}_4/\mathrm{CFT}_3$ precision match.

- The $D = 4$ magnetic coupling $m \equiv gc$, the $D = 10$ Romans mass $\hat{F}_{(0)}$ and the CS level k are related by

$$m = \hat{F}_{(0)} = k/(2\pi \ell_s) \ ,$$

where $\ell_s = \sqrt{\alpha'}$ is the string length.

Outline

1. Three puzzles
2. Dyonic ISO(7) supergravity
3. Consistent truncation of massive type IIA on S^6
4. AdS_4 solutions of massive IIA
5. $D = 11$ on S^7

Dyonic ISO(7) supergravity

The $D = 4$ $\mathcal{N} = 8$ tensor hierarchy

The bosonic fields of $D = 4$ $\mathcal{N} = 8$ supergravity come in irreps of $E_{7(7)}$. The p-forms, $p = 1, 2, 3, 4$, generate a 'tensor hierarchy'. [de Wit, Nicolai, Samtleben '08]

$\mathbf{1}$	metric :	ds_4^2
$\mathbf{56}$	coset representatives :	$\mathcal{V}_{\mathbb{M}}{}^{ij}$,
$\mathbf{56}$	vectors :	$\mathcal{A}^{\mathbb{M}}$,
$\mathbf{133}$	two-forms :	$\mathcal{B}_\alpha$,
$\mathbf{912}$	three-forms :	$\mathcal{C}_\alpha{}^{\mathbb{M}}$,
$\mathbf{133} + \mathbf{8645}$	four-forms	

The $D = 4$ $\mathcal{N} = 8$ duality hierarchy

The higher-rank forms carry dynamical degrees of freedom, albeit not independent ones. They can be expressed in terms of the lower-rank forms, scalars and metric via the 'duality hierarchy' [Bergshoeff, Hartong, Hohm, Huebscher, Ortin '09]

$$
\begin{aligned}
\tilde{\mathcal{H}}_{(2)\Lambda} &= \mathcal{R}_{\Lambda\Sigma}\,\mathcal{H}^{\Sigma}_{(2)} - \mathcal{I}_{\Lambda\Sigma} * \mathcal{H}^{\Sigma}_{(2)} \,, \\
\mathcal{H}_{(3)\alpha} &= -\tfrac{1}{12}\,(t_\alpha)_{\mathbb{M}}{}^{\mathbb{P}}\,\mathcal{M}_{\mathbb{N}\mathbb{P}} * D\mathcal{M}^{\mathbb{M}\mathbb{N}} \,, \\
\mathcal{H}_{(4)\alpha}{}^{\mathbb{M}} &= -\tfrac{1}{84}\,(t_\alpha)_{\mathbb{P}}{}^{\mathbb{R}} X_{\mathbb{N}\mathbb{Q}}{}^{\mathbb{S}} \mathcal{M}^{\mathbb{M}\mathbb{N}} \left(\mathcal{M}^{\mathbb{P}\mathbb{Q}} \mathcal{M}_{\mathbb{R}\mathbb{S}} + 7\,\delta^{\mathbb{P}}_{\mathbb{S}}\,\delta^{\mathbb{Q}}_{\mathbb{R}} \right) \mathrm{vol}_4 \,.
\end{aligned}
$$

The embedding tensor

In $\mathcal{N} = 8$ supergravity, all effects of the gauging are codified in the embedding tensor $\Theta_{\mathbb{M}}{}^{\alpha}$.

[de Wit, Samtleben, Trigiante '07]

This is subject to

- quadratic constraints, which ensure the consistency of the gauging, and
- linear constraints, which restrict it to the **912** of $E_{7(7)}$.

The dyonic ISO(7) embedding tensor

- To formulate the ISO(7) gauging, it is natural to branch out $E_{7(7)}$ into SL(7), since

$$\mathrm{ISO}(7) \equiv \mathrm{SO}(7) \ltimes \mathbb{R}^7 \subset \mathrm{SL}(7) \ltimes \mathbb{R}^7 \subset \mathrm{GL}(7) \ltimes \mathbb{R}^7 \subset \mathrm{SL}(8) \subset \mathrm{E}_{7(7)} \ .$$

- The embedding tensor of dyonic ISO(7) supergravity takes values in the $\mathbf{28}+\mathbf{1}$ of SL(7): [Dall'Agata, Inverso, '11]

$$\Theta_{[AB]}{}^{C}{}_{D} = 2\,\delta^{C}_{[A}\theta_{B]D} \quad , \quad \Theta^{[AB]C}{}_{D} = 2\,\delta^{[A}_{D}\xi^{B]C} \ .$$

  where

$$\theta = \mathrm{diag}(\mathbb{I}_7, 0) \ , \quad \xi = \mathrm{diag}(0_7, 1) \ ,$$

- The ISO(7)-covariant derivatives that follow from this are

$$D = d - g\,\mathcal{A}^{IJ}\,t_{[I}{}^{K}\,\delta_{J]K} + \left(g\,\delta_{IJ}\,\mathcal{A}^{I} - m\,\tilde{\mathcal{A}}_J\right) t_8{}^{J} \ .$$

A restricted duality hierarchy

For the ISO(7) gauging, a restricted duality hierarchy can be identified, still $\mathcal{N} = 8$ but only SL(7)-covariant:

$$
\begin{array}{rcl}
\mathbf{1} & \text{metric :} & ds_4^2 \\
\mathbf{21}' + \mathbf{7}' + \mathbf{21} + \mathbf{7} & \text{coset representatives :} & \mathcal{V}^{IJ\,ij} \,,\ \mathcal{V}^{I8\,ij} \,,\ \tilde{\mathcal{V}}_{IJ}{}^{ij} \,,\ \tilde{\mathcal{V}}_{I8}{}^{ij} \,, \\
\mathbf{21}' + \mathbf{7}' + \mathbf{21} + \mathbf{7} & \text{vectors :} & \mathcal{A}^{IJ} \,,\quad \mathcal{A}^{I} \,,\quad \tilde{\mathcal{A}}_{IJ} \,,\quad \tilde{\mathcal{A}}_{I} \,, \\
\mathbf{48} + \mathbf{7}' & \text{two-forms :} & \mathcal{B}_I{}^J \,,\quad \mathcal{B}^I \,, \\
\mathbf{28}' & \text{three-forms :} & \mathcal{C}^{IJ} \,.
\end{array}
$$

A restricted duality hierarchy

This restricted duality hierarchy has closed supersymmetry transformations and field equations. For example, the Bianchi identities close into

$$D\mathcal{H}^{IJ}_{(2)} = 0 \quad , \quad D\mathcal{H}^{I}_{(2)} = m\,\mathcal{H}^{I}_{(3)} \quad , \quad D\tilde{\mathcal{H}}_{(2)IJ} = -2\,g\,\mathcal{H}_{(3)[I}{}^{K}\,\delta_{J]K} \quad , \quad D\tilde{\mathcal{H}}_{(2)I} = g\,\delta_{IJ}\,\mathcal{H}^{J}_{(3)} \;,$$

$$D\mathcal{H}_{(3)I}{}^{J} = \mathcal{H}^{JK}_{(2)} \wedge \tilde{\mathcal{H}}_{(2)IK} + \mathcal{H}^{J}_{(2)} \wedge \tilde{\mathcal{H}}_{(2)I} - 2g\,\delta_{IK}\,\mathcal{H}^{JK}_{(4)} - \tfrac{1}{7}\,\delta^{J}_{I}\,(\text{trace}) \;,$$

$$D\mathcal{H}^{I}_{(3)} = -\mathcal{H}^{IJ}_{(2)} \wedge \tilde{\mathcal{H}}_{(2)J} \quad , \qquad D\mathcal{H}^{IJ}_{(4)} \equiv 0 \;.$$

A restricted duality hierarchy

The duality relations close into

$$\tilde{\mathcal{H}}_{(2)IJ} = -\tfrac{1}{2}\mathcal{I}_{[IJ][KL]} * \mathcal{H}^{KL}_{(2)} - \mathcal{I}_{[IJ][K8]} * \mathcal{H}^{K}_{(2)} + \tfrac{1}{2}\mathcal{R}_{[IJ][KL]}\,\mathcal{H}^{KL}_{(2)} + \mathcal{R}_{[IJ][K8]}\,\mathcal{H}^{K}_{(2)} \,,$$

$$\tilde{\mathcal{H}}_{(2)I} = -\tfrac{1}{2}\mathcal{I}_{[I8][KL]} * \mathcal{H}^{KL}_{(2)} - \mathcal{I}_{[I8][K8]} * \mathcal{H}^{K}_{(2)} + \tfrac{1}{2}\mathcal{R}_{[I8][KL]}\,\mathcal{H}^{KL}_{(2)} + \mathcal{R}_{[I8][K8]}\,\mathcal{H}^{K}_{(2)} \,,$$

$$\mathcal{H}_{(3)I}{}^{J} = -\tfrac{1}{12}(t_I{}^J)_{\mathbb{M}}{}^{\mathbb{P}}\,\mathcal{M}_{\mathbb{NP}} * D\mathcal{M}^{\mathbb{MN}} - \tfrac{1}{7}\,\delta^J_I\,(\text{trace})\,,$$

$$\mathcal{H}_{(3)}{}^{I} = -\tfrac{1}{12}(t_8{}^I)_{\mathbb{M}}{}^{\mathbb{P}}\,\mathcal{M}_{\mathbb{NP}} * D\mathcal{M}^{\mathbb{MN}}\,,$$

$$\mathcal{H}^{IJ}_{(4)} = \tfrac{1}{84}X_{\mathbb{NQ}}{}^{\mathbb{S}}\big((t_K{}^{(I|})_{\mathbb{P}}{}^{\mathbb{R}}\mathcal{M}^{|J)K\,\mathbb{N}} + (t_8{}^{(I|})_{\mathbb{P}}{}^{\mathbb{R}}\mathcal{M}^{|J)8\,\mathbb{N}}\big)\big(\mathcal{M}^{\mathbb{PQ}}\mathcal{M}_{\mathbb{RS}} + 7\,\delta^{\mathbb{P}}_{\mathbb{S}}\,\delta^{\mathbb{Q}}_{\mathbb{R}}\big)\text{vol}_4\,.$$

Supersymmetric critical points

The dyonic ISO(7) gauging displays a rich structure of critical points, both supersymmetric and non-supersymmetric, all of them AdS. In contrast, the purely electric gauging has no known vacua.

SUSY	bos. sym.	ref.
$\mathcal{N}=3$	SO(4)	[Gallerati, Samtleben, Trigiante '14]
$\mathcal{N}=2$	$SU(3)\times U(1)$	[Guarino, Jafferis, OV '15]
$\mathcal{N}=1$	G_2	[Borghese, Guarino, Roest '12]
$\mathcal{N}=1$	SU(3)	[Guarino, OV '15]

Consistent truncation of massive type IIA on S^6

The strategy

In order to find the full embedding of dyonic ISO(7) supergravity in massive IIA, we follow two steps:

- We adapt the de Wit-Nicolai $D = 11$ approach to IIA. The IIA bosonic and fermionic field content and supersymmetry transformations are rewritten with $\mathrm{SO}(1,3) \times \mathrm{SL}(7)$ and $\mathrm{SO}(1,3) \times \mathrm{SU}(8)$ covariance.

- We develop a new technique: exploit the $D = 4$ restricted duality hierarchy.

Type IIA with only SO(1,3) manifest

Under

$$\mathrm{SO}(1,9) \to \mathrm{SO}(1,3) \times \mathrm{SO}(6) \ ,$$

the IIA fields split as

$$d\hat{s}^2_{10} = \Delta^{-1}\, ds^2_4 + g_{mn}\big(dy^m + B^m\big)\big(dy^n + B^n\big) \ ,$$

$$\begin{aligned} \hat{A}_{(3)} &= \tfrac{1}{6} A_{\mu\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho + \tfrac{1}{2} A_{\mu\nu m}\, dx^\mu \wedge dx^\nu \wedge \big(dy^m + B^m\big) \\ &+ \tfrac{1}{2} A_{\mu m n}\, dx^\mu \wedge \big(dy^m + B^m\big) \wedge \big(dy^n + B^n\big) \\ &+ \tfrac{1}{6} A_{mnp}\big(dy^m + B^m\big) \wedge \big(dy^n + B^n\big) \wedge \big(dy^p + B^p\big) \ , \end{aligned}$$

$$\hat{B}_{(2)} = \tfrac{1}{2} B_{\mu\nu}\, dx^\mu \wedge dx^\nu + B_{\mu m}\, dx^\mu \wedge \big(dy^m + B^m\big) + \tfrac{1}{2} B_{mn}\big(dy^m + B^m\big) \wedge \big(dy^n + B^n\big) \ ,$$

$$\hat{A}_{(1)} = A_\mu\, dx^\mu + A_m\big(dy^m + B^m\big) \ ,$$

Type IIA with only SO(1,3) manifest

- SO(6) can by straightforwardly promoted to SL(6). Then we have the SO(1,3)-covariant field content in SL(6) representations:

$$
\begin{array}{rcl}
\mathbf{1} & \text{metric :} & ds_4^2 \ , \\
\mathbf{21}+\mathbf{6}+\mathbf{1}+\mathbf{20}+\mathbf{15} & \text{scalars :} & g_{mn} \ , \ A_m \ , \ \hat{\phi} \ , \ A_{mnp} \ , \ B_{mn} \ , \\
\mathbf{6}'+\mathbf{1}+\mathbf{15}+\mathbf{6} & \text{vectors :} & B_\mu{}^m \ , \ A_\mu \ , \ A_{\mu mn} \ , \ B_{\mu m} \ , \\
\mathbf{6}+\mathbf{1} & \text{two-forms :} & A_{\mu\nu m} \ , \ B_{\mu\nu} \ , \\
\mathbf{1} & \text{three-form :} & A_{\mu\nu\rho} \ .
\end{array}
$$

- These can be grouped up into SL(7) irreps, too.

Non-linear redefinitions

Further redefinitions are needed so that the p-forms comply with the $D = 4$ transformations dictated by the tensor hierarchy:

- Vectors:

$$C_\mu{}^{m8} \equiv B_\mu{}^m \, , \qquad C_\mu{}^{78} \equiv A_\mu \, , \qquad \tilde{C}_{\mu\, mn} \equiv A_{\mu mn} - A_\mu B_{mn} \, , \qquad \tilde{C}_{\mu\, m7} \equiv B_{\mu m} \, ,$$

- Two-forms:

$$C_{\mu\nu\, m} \equiv -A_{\mu\nu m} + C_{[\mu}{}^{n8}\, \tilde{C}_{\nu]nm} + C_{[\mu}{}^{78}\, \tilde{C}_{\nu]m7} \, , \qquad C_{\mu\nu\, 7} \equiv -B_{\mu\nu} + C_{[\mu}{}^{m8}\, \tilde{C}_{\nu]m7} \, ,$$

- Three-form:

$$C_{\mu\nu\rho} \equiv A_{\mu\nu\rho} - C_{[\mu}{}^{m8}\, C_\nu{}^{n8}\, \tilde{C}_{\rho]mn} + C_{[\mu}{}^{m8}\, C_\nu{}^{78}\, \tilde{C}_{\rho]m7} + 3\, C_{[\mu}{}^{78} C_{\nu\rho]7} \, .$$

Similar redefinitions were previously considered in type IIB. [Ciceri, de Wit, OV '14; Samtleben, Hohm '15]

KK ansatz and consistency of the truncation

The KK ansatz naturally relates the SL(6)-covariant IIA field content to the restricted tensor hierarchy for the ISO(7) gauging and quantities on S^6:

- Vectors:

$$C_\mu{}^{m8}(x,y) = \tfrac{1}{2}\, g\, K^m_{IJ}(y)\, \mathcal{A}_\mu{}^{IJ}(x) \quad , \quad C_\mu{}^{78}(x,y) = -\mu_I(y)\, \mathcal{A}_\mu{}^I(x) \ ,$$

$$\tilde{C}_{\mu\, mn}(x,y) = \tfrac{1}{4}\, K^{IJ}_{mn}(y)\, \tilde{\mathcal{A}}_{\mu\, IJ}(x) \quad , \quad \tilde{C}_{\mu\, m7}(x,y) = -g^{-1}\, (\partial_m \mu^I)(y)\, \tilde{\mathcal{A}}_{\mu\, I}(x) \ ,$$

- Two-forms:

$$C_{\mu\nu\, m}(x,y) = -g^{-1}\, (\mu_I \partial_m \mu^J)(y)\, \mathcal{B}_{\mu\nu\, J}{}^I(x) \quad , \quad C_{\mu\nu\, 7}(x,y) = \mu_I(y)\, \mathcal{B}_{\mu\nu}{}^I(x) \ .$$

- Three-form:

$$C_{\mu\nu\rho}(x,y) = (\mu_I \mu_J)(y)\, \mathcal{C}_{\mu\nu\rho}{}^{IJ}(x) \ ,$$

- Similarly with the metric, scalars and fermions.

When these ansatze are introduced into the SL(7)-covariant IIA susy transformations, the S^6 dependence drops out and the susy transformations of the restricted $D=4$ hierarchy arise.

The full non-linear embedding

$$d\hat{s}_{10}^2 = \Delta^{-1}\, ds_4^2 + g_{mn}\, Dy^m\, Dy^n \ ,$$

$$\begin{aligned}\hat{A}_{(3)} &= \mu_I\mu_J\,(\mathcal{C}^{IJ} + \mathcal{A}^I\wedge\mathcal{B}^J + \tfrac{1}{6}\mathcal{A}^{IK}\wedge\mathcal{A}^{JL}\wedge\tilde{\mathcal{A}}_{KL} + \tfrac{1}{6}\mathcal{A}^I\wedge\mathcal{A}^{JK}\wedge\tilde{\mathcal{A}}_K) \\ &\quad + g^{-1}\,(\mathcal{B}_J{}^I + \tfrac{1}{2}\mathcal{A}^{IK}\wedge\tilde{\mathcal{A}}_{KJ} + \tfrac{1}{2}\mathcal{A}^I\wedge\tilde{\mathcal{A}}_J)\wedge\mu_I D\mu^J + \tfrac{1}{2}g^{-2}\,\tilde{\mathcal{A}}_{IJ}\wedge D\mu^I\wedge D\mu^J \\ &\quad - \tfrac{1}{2}\,\mu_I\, B_{mn}\,\mathcal{A}^I\wedge Dy^m\wedge Dy^n + \tfrac{1}{6}A_{mnp}\, Dy^m\wedge Dy^n\wedge Dy^p \ ,\end{aligned}$$

$$\hat{B}_{(2)} = -\mu_I\,(\mathcal{B}^I + \tfrac{1}{2}\mathcal{A}^{IJ}\wedge\tilde{\mathcal{A}}_J) - g^{-1}\,\tilde{\mathcal{A}}_I\wedge D\mu^I + \tfrac{1}{2}B_{mn}\, Dy^m\wedge Dy^n \ ,$$

$$\hat{A}_{(1)} = -\mu_I\,\mathcal{A}^I + A_m\, Dy^m \ .$$

Here, the covariant derivatives are

$$Dy^m \equiv dy^m + \tfrac{1}{2}\, g\, K^m_{IJ}\,\mathcal{A}^{IJ} \quad , \quad D\mu^I \equiv d\mu^I - g\,\mathcal{A}^{IJ}\mu_J \ .$$

The full non-linear embedding

The internal (inverse) metric and forms are given in terms of SL(7)-covariant blocks of the $D=4$ scalar matrix $\mathcal{M}_{\mathbb{MN}}$ and S^6 quantities as

$$
\begin{aligned}
g^{mn} &= \tfrac{1}{4} g^2 \, \Delta \, K^m_{IJ} \, K^n_{KL} \, \mathcal{M}^{IJ\,KL} \, , \\
A_m &= \tfrac{1}{2} \, g \, \Delta \, g_{mn} \, K^n_{IJ} \, \mu_K \, \mathcal{M}^{IJ\,K8} \, , \\
B_{mn} &= -\tfrac{1}{2} \, \Delta \, g_{mp} \, K^p_{IJ} \, \partial_n \mu^K \, \mathcal{M}^{IJ}{}_{K8} \, , \\
A_{mnp} &= A_m B_{np} + \tfrac{1}{8} \, g \, \Delta \, g_{mq} \, K^q_{IJ} \, K^{KL}_{np} \mathcal{M}^{IJ}{}_{KL} .
\end{aligned}
$$

Field strengths

The embedding can be given in terms of independent $D = 4$ degrees of freedom.

- Compute the field strengths using their type IIA definitions:

$$\hat{F}_{(4)} = \mu_I \mu_J \, \mathcal{H}^{IJ}_{(4)} + g^{-1} \, \mathcal{H}_{(3)\,J}{}^{I} \wedge \mu_I D\mu^J + \tfrac{1}{2} \, g^{-2} \, \tilde{\mathcal{H}}_{(2)IJ} \wedge D\mu^I \wedge D\mu^J + \dots \, ,$$

$$\hat{H}_{(3)} = -\mu_I \, \mathcal{H}^I_{(3)} - g^{-1} \, \tilde{\mathcal{H}}_{(2)I} \wedge D\mu^I + \dots \, ,$$

$$\hat{F}_{(2)} = -\mu_I \mathcal{H}^I_{(2)} + g^{-1} \left(g \, \delta_{IJ} \, \mathcal{A}^J - m \, \tilde{A}_I \right) \wedge D\mu^I + \dots \, ,$$

Here, $\mathcal{H}^{IJ}_{(4)}$, etc., turn out to be the field strengths of the restricted $D = 4$ duality hierarchy.

- These field strengths are now regarded as short-hand for the corresponding $D = 4$ dualised expressions.

The Freund-Rubin term

- An elegant expression can be found for the Freund Rubin term using the duality condition for $\mathcal{H}^{IJ}_{(4)}$. It can be written as $\hat{F}_{(4)} = U\,\mathrm{vol}_4$, where

$$U = -\frac{g}{84}\, X'_{\mathbb{MP}}{}^{\mathbb{R}} X_{\mathbb{NQ}}{}^{\mathbb{S}} \mathcal{M}^{\mathbb{MN}} \Big(\mathcal{M}^{\mathbb{PQ}} \mathcal{M}_{\mathbb{RS}} + 7\, \delta^{\mathbb{P}}_{\mathbb{S}}\, \delta^{\mathbb{Q}}_{\mathbb{R}} \Big)$$

closely parallels the scalar potential of $D = 4$ $\mathcal{N} = 8$ gauged supergravity

$$V = \frac{g^2}{168}\, X_{\mathbb{MP}}{}^{\mathbb{R}} X_{\mathbb{NQ}}{}^{\mathbb{S}} \mathcal{M}^{\mathbb{MN}} \Big(\mathcal{M}^{\mathbb{PQ}} \mathcal{M}_{\mathbb{RS}} + 7\, \delta^{\mathbb{P}}_{\mathbb{S}}\, \delta^{\mathbb{Q}}_{\mathbb{R}} \Big) \, .$$

- $X'_{\mathbb{MN}}{}^{\mathbb{P}} = \Theta'_{\mathbb{M}}{}^{\alpha}\, (t_\alpha)_{\mathbb{N}}{}^{\mathbb{P}}$ is defined in terms of an S^6-dependent $\Theta'_{\mathbb{M}}{}^{\alpha}$, with

$$\Theta'_{[AB]}{}^{C}{}_{D} = 2\, \delta^{C}_{[A}\, \theta'_{B]D} \qquad , \qquad \Theta'^{[AB]C}{}_{D} = 2\, \delta^{[A}_{D}\, \xi'^{B]C} \, ,$$

and

$$\theta'_{IJ} = \mu_I \mu_J \, , \quad \theta'_{I8} = 0 \, , \quad \theta'_{88} = 0 \, ; \qquad \xi'^{AB} = 0 \, .$$

A two-scalar subsector

$$d\hat{s}_{10}^2 = e^{\frac{1}{8}(2\phi-\varphi)}\Delta^{5/8}\Big[\, ds_4^2 + g^{-2}e^{-2\phi+\varphi}d\alpha^2 + g^{-2}\Delta^{-1}\sin^2\alpha\, d\tilde{s}^2(S^5)\Big] \,,$$

$$e^{\hat{\phi}} = e^{\frac{1}{4}(6\phi+5\varphi)}\Delta^{-1/4} \,,$$

$$\hat{A}_{(3)} = \cos^2\alpha\, C^0 + \sin^2\alpha\, C^1 - g^{-1}\sin\alpha\cos\alpha\, B_1\wedge d\alpha \,, \quad \hat{B}_{(2)} = 0 \,, \quad \hat{A}_{(1)} = 0 \,.$$

with

$$\Delta = e^{\varphi}\sin^2\alpha + e^{2\phi-\varphi}\cos^2\alpha \,.$$

The field strength is

$$\hat{F}_{(4)} = H^0_{(4)}\,\cos^2\alpha + H^1_{(4)}\,\sin^2\alpha - g^{-1}\sin\alpha\cos\alpha\, H_{(3)1}\wedge d\alpha \,.$$

And using the dualisation conditions:

$$\begin{aligned}\hat{F}_{(4)} \quad &= \quad g\Big[\big(6\,e^{2\phi-\varphi} - e^{4\phi-3\varphi}\big)\,\cos^2\alpha + \big(4\,e^{\varphi} + e^{2\phi-\varphi}\big)\,\sin^2\alpha\Big]\mathrm{vol}_4 \\ &\quad +2g^{-1}\sin\alpha\cos\alpha\, d\alpha\wedge *\big(d\varphi - d\phi\big) \,.\end{aligned}$$

AdS$_4$ solutions of massive IIA

Uplift of supersymmetric critical points

- By the consistency of the truncation, all solutions of the $D = 4$ supergravity give rise to solutions of massive type IIA. In particular, the (AdS) critical points uplift to $AdS_4 \times S^6$ solutions of massive IIA.

SUSY	bos. sym.	ref.
$\mathcal{N} = 3$	$SO(4)$	[Pang, Rong '15]
$\mathcal{N} = 2$	$SU(3) \times U(1)$	[Guarino, Jafferis, OV '15]
$\mathcal{N} = 1$	G_2	[Behrndt, Cvetic '04]
$\mathcal{N} = 1$	$SU(3)$	[OV '15]

An $\mathcal{N}=2$ AdS_4 solution of massive IIA

$$d\hat{s}^2_{10} = L^2 \left(3+\cos 2\alpha\right)^{1/2} \left(5+\cos 2\alpha\right)^{1/8} \Big[ds^2(\mathrm{AdS}_4) + \frac{3}{2}\, d\alpha^2 + \frac{6 \sin^2\alpha}{3+\cos 2\alpha}\, ds^2(\mathbb{CP}^2) + \frac{9\sin^2\alpha}{5+\cos 2\alpha}\, \boldsymbol{\eta}^2 \Big]$$

$$e^{\hat{\phi}} = e^{\phi_0} \frac{\left(5+\cos 2\alpha\right)^{3/4}}{3+\cos 2\alpha} \ ,$$

$$L^{-2} e^{-\frac{1}{2}\phi_0} \hat{H}_{(3)} = 24\sqrt{2}\, \frac{\sin^3\alpha}{\left(3+\cos 2\alpha\right)^2}\, \boldsymbol{J} \wedge d\alpha \ ,$$

$$L^{-3} e^{\frac{1}{4}\phi_0} \hat{F}_{(4)} = 6\,\mathrm{vol}(\mathrm{AdS}_4) + 12\sqrt{3}\, \frac{7+3\cos 2\alpha}{\left(3+\cos 2\alpha\right)^2}\, \sin^4\alpha \ \mathrm{vol}(\mathbb{CP}^2)$$

$$+18\sqrt{3}\, \frac{\left(9+\cos 2\alpha\right)\sin^3\alpha\cos\alpha}{\left(3+\cos 2\alpha\right)\left(5+\cos 2\alpha\right)}\, \boldsymbol{J}\wedge d\alpha \wedge \boldsymbol{\eta} \ ,$$

$$L^{-1} e^{\frac{3}{4}\phi_0} \hat{F}_{(2)} = -4\sqrt{6}\, \frac{\sin^2\alpha\cos\alpha}{\left(3+\cos 2\alpha\right)\left(5+\cos 2\alpha\right)}\, \boldsymbol{J} - 3\sqrt{6}\, \frac{\left(3-\cos 2\alpha\right)}{\left(5+\cos 2\alpha\right)^2}\, \sin\alpha \ d\alpha\wedge\boldsymbol{\eta} \ ,$$

Flux quantisation

- On our topologically S^6 solution, one can only impose quantisation conditions on $\hat{F}_{(0)}$ and $\hat{F}_{(6)}$:

$$k = 2\pi\ell_s \, \hat{F}_{(0)} \equiv 2\pi\ell_s \, m \,,$$

$$N = -\frac{1}{(2\pi\ell_s)^5} \int_{S^6} e^{\frac{1}{2}\hat{\phi}} \; \hat{*}\hat{F}_{(4)} + \hat{B}_{(2)} \wedge d\hat{A}_{(3)} + \frac{1}{6} m \, \hat{B}_{(2)} \wedge \hat{B}_{(2)} \wedge \hat{B}_{(2)}$$

- The classical parameters g and m (or L, e^{ϕ_0}) become fixed in terms of the quantum numbers k, N.

The dual CFT_3

- These backgrounds should arise as the near horizon of D2-branes on a smooth space with RR and NS fluxes.
- The presence of the Romans mass adds ($\mathcal{N} = 2$) Chern-Simons terms at level k to maximal 3D super-Yang-Mills with gauge group $SU(N)$.
- Three adjoint chirals with $\mathcal{W} = \text{Tr}(X[Y, Z])$, SU(3) flavour and U(1) R-symmetry.
- Free energy computed from the field theory and from the AdS dual. Find perfect agreement.

$D = 11$ on S^7

A restricted duality hierarchy

For the SO(8) gauging, a restricted duality hierarchy can be identified, still $\mathcal{N} = 8$ but only SL(8)-covariant:

$$
\begin{array}{rcl}
\mathbf{1} & \text{metric :} & ds_4^2 \\
\mathbf{28}' + \mathbf{28} & \text{coset representatives :} & \mathcal{V}^{IJ\,ij} \ , \ \tilde{\mathcal{V}}_{IJ}{}^{ij} \ , \quad , \\
\mathbf{28}' + \mathbf{28} & \text{vectors :} & \mathcal{A}^{IJ} \ , \quad \tilde{\mathcal{A}}_{IJ} \ , \\
\mathbf{63} & \text{two-forms :} & \mathcal{B}_I{}^J \ , \\
\mathbf{36}' & \text{three-forms :} & \mathcal{C}^{IJ} \ .
\end{array}
$$

This restricted duality hierarchy has closed supersymmetry transformations and field equations.

$D = 11$ supergravity with only $\mathrm{SO}(1,3)$ manifest

Under

$$\mathrm{SO}(1,10) \to \mathrm{SO}(1,3) \times \mathrm{SO}(7) \ ,$$

the IIA fields split as

$$\begin{aligned} d\hat{s}_{11}^2 &= \Delta^{-1}\, ds_4^2 + g_{mn}\big(dy^m + B^m\big)\big(dy^n + B^n\big) \ , \\ \hat{A}_{(3)} &= \tfrac{1}{6} A_{\mu\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho + \tfrac{1}{2} A_{\mu\nu m}\, dx^\mu \wedge dx^\nu \wedge \big(dy^m + B^m\big) \\ &\quad + \tfrac{1}{2} A_{\mu m n}\, dx^\mu \wedge \big(dy^m + B^m\big) \wedge \big(dy^n + B^n\big) \\ &\quad + \tfrac{1}{6} A_{mnp}\big(dy^m + B^m\big) \wedge \big(dy^n + B^n\big) \wedge \big(dy^p + B^p\big) \ , \end{aligned}$$

$D = 11$ supergravity with only SO$(1,3)$ manifest

SO(7) can by straightforwardly promoted to SL(7). Then we have the SO(1,3)-covariant field content in SL(7) representations:

$$
\begin{array}{rcl}
\mathbf{1} & \text{metric :} & ds_4^2 \ , \\
\mathbf{28} + \mathbf{35} & \text{scalars :} & g_{mn} \ , \ A_{mnp} \ , \\
\mathbf{7}' + \mathbf{21} & \text{vectors :} & B_\mu{}^m \ , \ A_{\mu mn} \ , \\
\mathbf{7} & \text{two-forms :} & A_{\mu\nu m} \ , \\
\mathbf{1} & \text{three-form :} & A_{\mu\nu\rho} \ .
\end{array}
$$

Non-linear redefinitions

Further redefinitions are needed so that the p-forms comply with the $D = 4$ transformations dictated by the tensor hierarchy:

- Vectors:

$$C_\mu{}^{m8} \equiv B_\mu{}^m \, , \qquad \tilde{C}_{\mu\, mn} \equiv A_{\mu mn} \, ,$$

- Two-forms:

$$C_{\mu\nu\, m} \equiv -A_{\mu\nu m} + C_{[\mu}{}^{n8}\, \tilde{C}_{\nu]nm} \, ,$$

- Three-form:

$$C_{\mu\nu\rho} \equiv A_{\mu\nu\rho} - C_{[\mu}{}^{m8}\, C_\nu{}^{n8}\, \tilde{C}_{\rho]mn} \, .$$

KK ansatz and consistency of the truncation

The KK ansatz naturally relates the SL(7)-covariant IIA field content to the restricted tensor hierarchy for the SO(8) gauging and quantities on S^7:

- Vectors:

$$C_\mu{}^{m8}(x,y) = \tfrac{1}{2}\, g\, K^m_{IJ}(y)\, \mathcal{A}_\mu{}^{IJ}(x) \qquad , \qquad \tilde{C}_{\mu\, mn}(x,y) = \tfrac{1}{4}\, K^{IJ}_{mn}(y)\, \tilde{\mathcal{A}}_{\mu\, IJ}(x)\ ,$$

- Two-forms:

$$C_{\mu\nu\, m}(x,y) = -g^{-1}\, (\mu_I \partial_m \mu^J)(y)\, \mathcal{B}_{\mu\nu\ J}{}^I(x)$$

- Three-form:

$$C_{\mu\nu\rho}(x,y) = (\mu_I \mu_J)(y)\, \mathcal{C}_{\mu\nu\rho}{}^{IJ}(x)\ ,$$

- Similarly with the metric, scalars and fermions.

When these ansatze are introduced into the SL(8)-covariant $D = 11$ susy transformations, the S^7 dependence drops out and the susy transformations of the restricted $D = 4$ hierarchy arise.

The full non-linear embedding

$$d\hat{s}^2_{11} = \Delta^{-1}\, ds^2_4 + \tfrac{1}{12}\, g^{-2}\, \Delta^2\, (t_I{}^J)_{\mathbb{M}}{}^{\mathbb{P}}\, (t_K{}^L)_{\mathbb{N}}{}^{\mathbb{Q}}\, \mathcal{M}^{\mathbb{M}\mathbb{N}}\, \mathcal{M}_{\mathbb{P}\mathbb{Q}}\, \mu_J \mu_L D\mu^I D\mu^K \ ,$$

$$\hat{A}_{(3)} = \mu_I \mu_J \left(\mathcal{C}^{IJ} + \tfrac{1}{6}\mathcal{A}^{IK} \wedge \mathcal{A}^{JL} \wedge \tilde{\mathcal{A}}_{KL}\right) + g^{-1} \left(\mathcal{B}_J{}^I + \tfrac{1}{2}\mathcal{A}^{IK} \wedge \tilde{\mathcal{A}}_{KJ}\right) \wedge \mu_I D\mu^J$$
$$+ \tfrac{1}{2} g^{-2}\, \tilde{\mathcal{A}}_{IJ} \wedge D\mu^I \wedge D\mu^J + A\ ,$$

with

$$A = -\tfrac{1}{72}\, g^{-3}\, \Delta^3\, (t_I{}^J)_{\mathbb{P}}{}^{\mathbb{R}}\, X'_{\mathbb{M}\mathbb{Q}}{}^{\mathbb{S}}\, \delta_{\mathbb{N}\mathbb{T}}\, \Omega^{\mathbb{T}\mathbb{U}}\, \Theta_{\mathbb{U}}{}^K{}_L\, \mathcal{M}^{\mathbb{M}\mathbb{N}}\, \mathcal{M}^{\mathbb{P}\mathbb{Q}}\, \mathcal{M}_{\mathbb{R}\mathbb{S}}\, \mu_J\, D\mu^I \wedge D\mu_K \wedge D\mu^L$$

and

$$\Delta^{-3} = \tfrac{1}{84}\, X'_{\mathbb{M}\mathbb{P}}{}^{\mathbb{R}}\, X'_{\mathbb{N}\mathbb{Q}}{}^{\mathbb{S}}\, \mathcal{M}^{\mathbb{M}\mathbb{N}} \left(\mathcal{M}^{\mathbb{P}\mathbb{Q}}\mathcal{M}_{\mathbb{R}\mathbb{S}} + 7\, \delta^{\mathbb{P}}_{\mathbb{S}}\, \delta^{\mathbb{Q}}_{\mathbb{R}}\right)\, .$$

Consistent KK truncations

$$\hat{F}_{(4)} = \mu_I \mu_J \, \mathcal{H}^{IJ}_{(4)} + g^{-1} \, \mathcal{H}_{(3)\,J}{}^{I} \wedge \mu_I D\mu^J + \tfrac{1}{2} \, g^{-2} \, \tilde{\mathcal{H}}_{(2)IJ} \wedge D\mu^I \wedge D\mu^J + dA \; .$$

Consistent KK truncations

$$\hat{F}_{(4)} = \mu_I \mu_J \mathcal{H}^{IJ}_{(4)} + g^{-1} \mathcal{H}_{(3)\,J}{}^I \wedge \mu_I D\mu^J + \tfrac{1}{2} g^{-2} \tilde{\mathcal{H}}_{(2)IJ} \wedge D\mu^I \wedge D\mu^J + dA \, .$$

$$\begin{aligned}
\hat{F}_{(4)} = \; & U \, \mathrm{vol}_4 + \tfrac{1}{12} g^{-1} (t_I{}^J)_{\mathbb{M}}{}^{\mathbb{P}} \mathcal{M}_{\mathbb{NP}} * D\mathcal{M}^{\mathbb{MN}} \wedge \mu_J D\mu^I \\
& + \tfrac{1}{4} g^{-2} \Big(\mathcal{I}_{[IJ][KL]} * \mathcal{H}^{KL}_{(2)} + \mathcal{R}_{[IJ][KL]} \mathcal{H}^{KL}_{(2)} \Big) \wedge D\mu^I \wedge D\mu^J \\
& - \tfrac{1}{24} g^{-2} \Delta^3 (t_I{}^J)_{\mathbb{P}}{}^{\mathbb{R}} X'_{\mathbb{MQ}}{}^{\mathbb{S}} \delta_{\mathbb{NT}} \Omega^{\mathbb{TU}} \Theta_{\mathbb{U}}{}^K{}_L \mathcal{M}^{\mathbb{MN}} \mathcal{M}^{\mathbb{PQ}} \mathcal{M}_{\mathbb{RS}} \delta_{KM} \mu_J \mu_N \mathcal{H}^{N[I}_{(2)} \wedge D\mu^M \wedge D\mu^{L]} \\
& - \tfrac{1}{6048} g^{-3} \Delta^6 (t_I{}^J)_{\mathbb{P}}{}^{\mathbb{R}} \delta_{\mathbb{NT}} \Omega^{\mathbb{TU}} \Theta_{\mathbb{U}}{}^K{}_L \\
& \qquad \times D\Big(X'_{\mathbb{MQ}}{}^{\mathbb{S}} X'_{\mathbb{VX}}{}^{\mathbb{Z}} X'_{\mathbb{WY}}{}^{\mathbb{A}} \mathcal{M}^{\mathbb{MN}} \mathcal{M}^{\mathbb{PQ}} \mathcal{M}^{\mathbb{VW}} \mathcal{M}^{\mathbb{XY}} \mathcal{M}_{\mathbb{RS}} \mathcal{M}_{\mathbb{ZA}} \Big) \wedge \mu_J D\mu^I \wedge D\mu_K \wedge D\mu^L \\
& - \tfrac{1}{72} g^{-3} \Delta^3 (t_I{}^J)_{\mathbb{P}}{}^{\mathbb{R}} X'_{\mathbb{MQ}}{}^{\mathbb{S}} \delta_{\mathbb{NT}} \Omega^{\mathbb{TU}} \Theta_{\mathbb{U}}{}^K{}_L \mathcal{M}^{\mathbb{MN}} \mathcal{M}^{\mathbb{PQ}} \mathcal{M}_{\mathbb{RS}} D\mu_J \wedge D\mu^I \wedge D\mu_K \wedge D\mu^L .
\end{aligned}$$

Thank you!