Cluster algebras from 2d gauge theories

Francesco Benini

Simons Center for Geometry and Physics
Stony Brook University

Texas A&M University
Heterotic Strings and (0,2) QFT
28 April 2014

with: D. Park, P. Zhao (to appear)

Introduction

2d gauge theories with $\mathcal{N} = (2, 2)$ supersymmetry

Seiberg-like dualities $\leftrightarrow$ Cluster Algebra structures

Introduction

2d gauge theories with $\mathcal{N} = (2,2)$ supersymmetry

Seiberg-like dualities $\leftrightarrow$ Cluster Algebra structures

Cluster algebra: [Fomin, Zelevinsky 2001] to describe coordinate rings of groups and Grassmannians

Other contexts:

- Teichmuller theory [Fock, Goncharov 03]
- Integrable systems (Y-systems)
- Wall crossing in 4d $\mathcal{N} = 2$ theories
- Amplitudes

Many interesting properties:

- Total positivity
- Laurent phenomenon
- Poisson structure
- ...

Cluster algebra [Fomin, Zelevinsky 01]

Commutative ring with unit and no zero divisors
with distinguished set of *generators* called cluster variables.

Set of *cluster variables* = (non-disjoint) union of
distinguished collections of n-subsets called clusters.

Cluster algebra [Fomin, Zelevinsky 01]

Commutative ring with unit and no zero divisors
with distinguished set of *generators* called cluster variables.

Set of *cluster variables* $=$ (non-disjoint) union of
distinguished collections of n-subsets called clusters.

- Exchange property (mutations):

for every cluster $\mathbf{x}$ and $x \in \mathbf{x}$,
there is another cluster obtained by substituting $x \to x'$ with rule

$$xx' = M_1 + M_2$$

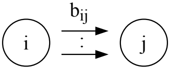
$M_{1,2}$: *monomials* in $n-1$ variables $\mathbf{x} \setminus \{x\}$, with no common divisors.

Any two clusters can be obtained from each other by sequence of *mutations*.

Cluster algebra [Fomin, Zelevinsky 06]

- Seed $\mathbf{x}$:

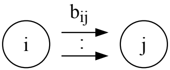
Skew-symmetric matrix b_{ij} $\quad\rightarrow\quad$ quiver B (no 1-, 2-cycles)
Coefficients $y_i \in \mathbb{P}$ *semifield* $(\cdot, \oplus)$ (*tropical*)
Cluster variables x_i.



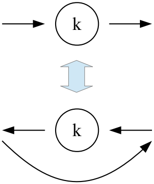
Cluster algebra [Fomin, Zelevinsky 06]

- Seed $\mathbf{x}$:

Skew-symmetric matrix b_{ij} $\rightarrow$ quiver B (no 1-, 2-cycles)
Coefficients $y_i \in \mathbb{P}$ *semifield* $(\cdot, \oplus)$ (*tropical*)
Cluster variables x_i.



- Mutation (at node k):



$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \\ b_{ij} + \operatorname{sign}(b_{ik})\,[b_{ik}b_{kj}]_+ & \text{otherwise} \end{cases}$$

$$y'_j = \begin{cases} y_k^{-1} & \text{if } j = k \\ y_j\, y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} & \text{otherwise} \end{cases}$$

$$x'_j = \begin{cases} \frac{1}{x_k}\left(\frac{y_k}{y_k \oplus 1} \prod_i x_i^{[b_{ik}]_+} + \frac{1}{y_k \oplus 1} \prod_i x_i^{[-b_{ik}]_+} \right) & \text{if } j = k \\ x_j & \text{otherwise .} \end{cases}$$

Hierarchical structure.

Cluster algebra

- Total positivity

  Cluster algebra transformations involve $+$, not $-$.

  Canonical choice of "positive" submanifold of a cluster manifold.

Cluster algebra

- Total positivity

  Cluster algebra transformations involve $+$, not $-$.

  Canonical choice of "positive" submanifold of a cluster manifold.

- Laurent phenomenon

  Any cluster variable x_i, viewed as a rational function of the variables in a given cluster $\mathbf{x}'$, is a Laurent polynomial.

  It is conjectured that has *positive* coefficients.

Cluster algebra

- Total positivity

  Cluster algebra transformations involve $+$, not $-$.

  Canonical choice of “positive” submanifold of a cluster manifold.

- Laurent phenomenon

  Any cluster variable x_i, viewed as a rational function of the variables in a given cluster $\mathbf{x}'$, is a Laurent polynomial.

  It is conjectured that has *positive* coefficients.

- Poisson structure

$$\{x_i, x_j\} = b_{ij} x_i x_j \qquad \text{extend by Liebniz} \qquad \text{(log canonical)}$$

  Such bracket is invariant under mutations.

Outline

- Seiberg-like dualities of 2d $\mathcal{N} = (2, 2)$ gauge theories
- S^2 partition function
- From dualities to cluster algebras
- (Speculative) applications

2d Seiberg-like dualities

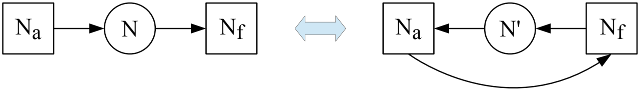
2d Seiberg-like dualities [Jockers, Kumar, Lapan, Morrison, Romo 12; FB, Cremonesi 12]

2d $\mathcal{N} = (2,2)$ SUSY gauge theories of vector and chiral multiplets

A: $U(N)$ with N_f fundamentals, N_a antifundamentals

$\updownarrow$

B: $U\big(\max(N_f, N_a) - N\big)$ with N_a fundamentals, N_f antifundamentals, $N_f N_a$ gauge singlets, superpotential $W_{\text{dual}} = \tilde{q} M q$



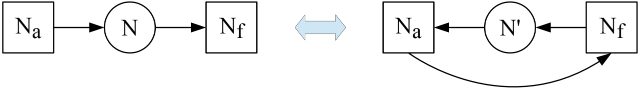
2d Seiberg-like dualities [Jockers, Kumar, Lapan, Morrison, Romo 12; FB, Cremonesi 12]

2d $\mathcal{N} = (2,2)$ SUSY gauge theories of vector and chiral multiplets

A: $U(N)$ with N_f fundamentals, N_a antifundamentals

$\updownarrow$

B: $U\big(\max(N_f, N_a) - N\big)$ with N_a fundamentals, N_f antifundamentals, $N_f N_a$ gauge singlets, superpotential $W_{\text{dual}} = \tilde{q} M q$



Comments:

- cfr. with 4d: no gauge anomaly $\to$ any N_f, N_a
- similar to Hori-Tong duality – but $U(N)$ instead of $SU(N)$
- $N_f = N_a$: flow to IR CFT (otherwise gapped)
- deformations: complexified FI term, twisted masses, superpotential terms

$$t = 2\pi\xi + i\theta \;, \qquad m_j \;, \qquad \tilde{m}_f \qquad \Rightarrow \qquad z \simeq e^{-t} \;, \qquad \boxed{z = \frac{1}{z'}}$$

Geometric interpretation [Jia, Sharpe, Wu 14]

Large positive FI (assume $N_f \geq N_a$): geometric realization

$$\mathsf{Gr}(N, N_f) \qquad = \qquad \mathsf{Gr}(N_f - N, N_F)$$

- Theory A: each antifundamental gives a copy of tautological bundle S

Equality of bundles:

$$S^{N_a} \to \mathsf{Gr}(N, N_f) \qquad = \qquad (Q^*)^{N_a} \to \mathsf{Gr}(N_f - N, N_f)$$

Universal quotient bundle: $\qquad 0 \to S \to \mathcal{O}^{N_f} \to Q \to 0$

- Theory B: mesons give $(\mathcal{O}^{N_f})^{N_a}$, superpotential imposes short exact sequence.

Chiral ring

Chiral operators modulo F-term relations

Theory A: $\tilde{Q}_f Q_j$ (no baryons)

Theory B: $\tilde{q}_j q_f$, M_{fj}, but W imposes $\tilde{q}_j q_f = 0$

Map: $\tilde{Q}_f Q_j = M_{fj}$

No quantum corrections in the chiral ring

- Add superpotential $W = f(\tilde{Q}_f Q_j) = f(M_{fj})$

Twisted chiral ring

Generators: $\mathrm{Tr}\,\sigma^k$ $k = 1, \ldots, N$ or symm. polynomials in $\sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_N)$

$$Q(x) = \det(x - \sigma) = x^N - x^{N-1}\,\mathrm{Tr}\,\sigma + \ldots$$

Twisted chiral ring

Generators: $\mathrm{Tr}\,\sigma^k$ $k = 1, \ldots, N$ or symm. polynomials in $\sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_N)$

$$Q(x) = \det(x - \sigma) = x^N - x^{N-1}\,\mathrm{Tr}\,\sigma + \ldots$$

Relations: effective superpotential on the Coulomb branch $\qquad t = 2\pi\xi + i\theta$

$$\widetilde{W}_{\text{eff}} = -t\sum_a \sigma_a - \sum_j \sum_{\rho\in R_j} \big(\rho(\sigma) - m_j\big)\big[\log\big(\rho(\sigma) - m_j\big) - 1\big]$$

Impose $0 = \partial\widetilde{W}_{\text{eff}}/\partial\sigma_a$ and $\sigma_a \neq \sigma_b$: $\qquad$ (*quantum equivariant coh*) $\qquad z \simeq e^{-t}$

$$\boxed{\prod_{j=1}^{N_f}(x - m_f) + i^{N_a - N_f}\, z \prod_{f=1}^{N_a}(x - \tilde{m}_f) = C(z)\, Q(x)\, T(x)}$$

$T(x)$ has degree $N' = \max(N_f, N_a) - N$

$$C(z) = \begin{cases} 1 & \text{if } N_f > N_a \\ 1 + z & \text{if } N_f = N_a \\ i^{N_a - N_f}\, z & \text{if } N_f < N_a \end{cases}$$

Twisted chiral ring

Generators: $\mathrm{Tr}\,\sigma^k$ $k = 1, \dots, N$ or symm. polynomials in $\sigma = \text{diag}(\sigma_1, \dots, \sigma_N)$

$$Q(x) = \det(x - \sigma) = x^N - x^{N-1}\,\mathrm{Tr}\,\sigma + \dots$$

Relations: effective superpotential on the Coulomb branch $\qquad t = 2\pi\xi + i\theta$

$$\widetilde{W}_{\text{eff}} = -t \sum_a \sigma_a - \sum_j \sum_{\rho \in R_j} \big(\rho(\sigma) - m_j\big)\big[\log\big(\rho(\sigma) - m_j\big) - 1\big]$$

Impose $0 = \partial \widetilde{W}_{\text{eff}} / \partial \sigma_a$ and $\sigma_a \neq \sigma_b$: $\qquad$ (*quantum equivariant coh*) $\qquad z \simeq e^{-t}$

$$\boxed{\prod_{j=1}^{N_f} (x - m_f) + i^{N_a - N_f} z \prod_{f=1}^{N_a} (x - \tilde{m}_f) = C(z)\, Q(x)\, T(x)}$$

$T(x)$ has degree $N' = \max(N_f, N_a) - N$

$$C(z) = \begin{cases} 1 & \text{if } N_f > N_a \\ 1 + z & \text{if } N_f = N_a \\ i^{N_a - N_f} z & \text{if } N_f < N_a \end{cases}$$

- Duality: $T(x) = Q'(x) = \det(x - \sigma')$

S^2 partition function [FB, Cremonesi 12; Doroud, Gomis, Le Floch, Lee 12]

Any Euclidean 2d $\mathcal{N} = (2,2)$ theory with R_V-symmetry can be placed supersymmetrically on S^2, with *no twist*.

$$Z_{S^2}(\text{param}) = \int \mathcal{D}\phi \, e^{-S(\text{param})}$$

Parameters:

- complexified FI term $\quad t = 2\pi\xi + i\theta, \quad z \simeq e^{-t}$
- real twisted masses m_j, $\tilde{m}_f$ and flavor magnetic fluxes $\mathfrak{n}_j$, $\tilde{\mathfrak{n}}_f$
- R-charges R

S^2 partition function [FB, Cremonesi 12; Doroud, Gomis, Le Floch, Lee 12]

Any Euclidean 2d $\mathcal{N}=(2,2)$ theory with R_V-symmetry can be placed supersymmetrically on S^2, with *no twist*.

$$Z_{S^2}(\text{param}) = \int \mathcal{D}\phi \, e^{-S(\text{param})}$$

Parameters:

- complexified FI term $\quad t = 2\pi\xi + i\theta, \qquad z \simeq e^{-t}$
- real twisted masses m_j, $\tilde{m}_f$ and flavor magnetic fluxes $\mathfrak{n}_j$, $\tilde{\mathfrak{n}}_f$
- R-charges R

Localization:

$$Z_{S^2} = \frac{1}{|\mathcal{W}|} \sum_{\mathfrak{m}\in\mathbb{Z}^N} \int d^N\sigma \, e^{i\,\mathbb{R}\mathrm{e}\,\widetilde{W}\left(\sigma+\frac{i}{2}\mathfrak{m}\right)} \, Z_{\text{1-loop}}$$

$$Z^{\text{gauge}}_{\text{1-loop}} = \prod_{\text{roots } \alpha>0} \left(\frac{\alpha(\mathfrak{m})^2}{4} + \alpha(\sigma)^2 \right)$$

$$Z^{\text{chiral}}_{\text{1-loop}} = \prod_{\text{chiral } \Phi} \prod_{\rho\in\mathfrak{R}_j} \frac{\Gamma\left(\frac{R[\Phi]}{2} - i\rho(\sigma) - if^i[\Phi]m_i - \frac{\rho(\mathfrak{m})+f^i[\Phi]\mathfrak{n}_i}{2}\right)}{\Gamma\left(1 - \frac{R[\Phi]}{2} + i\rho(\sigma) + if^i[\Phi]m_i - \frac{\rho(\mathfrak{m})+f^i[\Phi]\mathfrak{n}_i}{2}\right)}$$

S^2 partition function

We can prove:

$$Z^{(A)}\big(m_j, \mathfrak{n}_j, \tilde{m}_f, \tilde{\mathfrak{n}}_f; z\big) = f_{\text{contact}}\ \ f_{\text{Kt}}\ \ Z^{(B)}\Big(\tilde{m}_f - \tfrac{i}{2}, \tilde{\mathfrak{n}}_f, m_j - \tfrac{i}{2}, \tilde{\mathfrak{n}}_f; z^{-1}\Big)$$

Method: vortex partition function

$$Z_{S^2} = \sum_{\text{Higgs vacua}} Z_{\text{cl}}\ Z'_{\text{1-loop}}\ Z_{\text{vortex}}\ Z_{\text{antivortex}}$$

S^2 partition function

We can prove:

$$Z^{(A)}\big(m_j, \mathfrak{n}_j, \tilde{m}_f, \tilde{\mathfrak{n}}_f; z\big) = f_{\text{contact}}\ \ f_{\text{Kt}}\ \ Z^{(B)}\Big(\tilde{m}_f - \tfrac{i}{2}, \tilde{\mathfrak{n}}_f, m_j - \tfrac{i}{2}, \tilde{\mathfrak{n}}_f; z^{-1}\Big)$$

Method: vortex partition function

$$Z_{S^2} = \sum_{\text{Higgs vacua}} Z_{\text{cl}}\ Z'_{\text{1-loop}}\ Z_{\text{vortex}}\ Z_{\text{antivortex}}$$

- Mass shift: shift of R-charges, compatible with W_{dual}

$$r \equiv R[\tilde{Q}_f Q_j] = R[M_{fj}] \qquad \Rightarrow \qquad R[\tilde{q}_j q_f] = 2 - r$$

- f_{Kt}: real function $\to$ Kähler transformation

FT: Lagrangian term, improvement transformation [Closset, Cremonesi 14]

Geom: according to [Jockers, Kumar, Lapan, Morrison, Romo 12]

$$Z_{S^2} = e^{-K_{\text{Kähler}}}$$

Kähler transformation (does not affect the metric)

The contact term

- $f_{\text{contact}} = e^{i\,\mathbb{R}\mathrm{e}\,\widetilde{W}_{\text{contact}}}$: phase

Twisted superpotential, only function of parameters (twisted chirals):

$$m_j + \tfrac{i}{2}\mathfrak{n}_j \;, \qquad \tilde{m}_f + \tfrac{i}{2}\tilde{\mathfrak{n}}_f \;, \qquad t$$

Theory A: $\widetilde{W}_A = -t \operatorname{Tr}\sigma = \log z \operatorname{Tr}\sigma$

Theory B: result depends on number of flavors

- $N_f > N_a + 1$: $\widetilde{W}_B = \log z^{-1} \operatorname{Tr}\sigma + \log z \operatorname{Tr} m$
- $N_f = N_a + 1$: $\widetilde{W}_B = \log z^{-1} \operatorname{Tr}\sigma + \log z \operatorname{Tr} m + iz$
- $N_f = N_a$: $\widetilde{W}_B = \log z^{-1} \operatorname{Tr}\sigma + \log \frac{z}{1+z} \operatorname{Tr} m + \log(1+z) \operatorname{Tr}\tilde{m}$

The contact term

- $f_{\text{contact}} = e^{i\,\mathbb{R}\mathrm{e}\,\widetilde{W}_{\text{contact}}}$: phase

Twisted superpotential, only function of parameters (twisted chirals):

$$m_j + \tfrac{i}{2}\mathfrak{n}_j\ , \qquad \tilde{m}_f + \tfrac{i}{2}\tilde{\mathfrak{n}}_f\ , \qquad t$$

Theory A: $\widetilde{W}_A = -t\,\mathrm{Tr}\,\sigma = \log z\,\mathrm{Tr}\,\sigma$

Theory B: result depends on number of flavors

- $N_f > N_a + 1$: $\widetilde{W}_B = \log z^{-1}\,\mathrm{Tr}\,\sigma + \log z\,\mathrm{Tr}\,m$
- $N_f = N_a + 1$: $\widetilde{W}_B = \log z^{-1}\,\mathrm{Tr}\,\sigma + \log z\,\mathrm{Tr}\,m + iz$
- $N_f = N_a$: $\widetilde{W}_B = \log z^{-1}\,\mathrm{Tr}\,\sigma + \log \frac{z}{1+z}\,\mathrm{Tr}\,m + \log(1+z)\,\mathrm{Tr}\,\tilde{m}$

If flavor symmetry is gauged, m, $\tilde{m} \to$ dynamical twisted chiral multiplets

$$\delta\widetilde{W} = \log z_f\,\mathrm{Tr}\,m + \log z_a\,\mathrm{Tr}\,\tilde{m}$$

f_{contact} transforms neighboring FIs.

Quivers

Quivers and cluster algebras

Class of quiver gauge theories: no 1-cycles nor 2-cycles.

Apply Seiberg-like duality to a node k

- Quiver: almost CA action

Never generate 1-cycles. But assume that all 2-cycles are accompanied by *quadratic* superpotential $\rightarrow$ integrate them out.

Quivers and cluster algebras

Class of quiver gauge theories: no 1-cycles nor 2-cycles.

Apply Seiberg-like duality to a node k

- Quiver: almost CA action

Never generate 1-cycles. But assume that all 2-cycles are accompanied by *quadratic* superpotential $\to$ integrate them out.

- Coefficients. Transformation of ranks $N' = \max(N_f, N_a) - N$

Tropical semifield $\mathbb{P}$ $(\cdot, \oplus)$: $u^a \cdot u^b = u^{a+b}, \qquad u^a \oplus u^b = u^{\max(a,b)}$
u is formal variable

Ranks: $r_i = u^{N_i}$

$$r'_j = \begin{cases} r_k^{-1}\left(\prod_i r_i^{[b_{ij}]_+} \oplus \prod_i r_i^{[-b_{ij}]_+}\right) & j = k \\ r_j & \text{otherwise} \end{cases}$$

Beta-functions: $y_j \equiv \prod_i r_i^{b_{ij}} \quad \Rightarrow \quad \boxed{y_j = u^{\beta_j}}$

y_j transform as CA coefficients. u interpreted as cutoff scale.

Cluster variables

Transformation of FI parameters $z \simeq e^{-t}$:

$$
\begin{array}{llll}
N_a < N_f : & z_a \to z_a & z \to z^{-1} & z_f \to z_f z \\
N_a = N_f : & z_a \to z_a(1+z) & z \to z^{-1} & z_f \to z_f \frac{z}{1+z} \\
N_a > N_f : & z_a \to z_a z & z \to z^{-1} & z_f \to z_f
\end{array}
$$

From cluster variables define: $z_j = \prod_i x_i^{b_{ij}}$

$$
z'_j = \begin{cases} z_k^{-1} & j = k \\ z_j z_k^{[b_{kj}]_+} \left(\frac{y_k}{y_k \oplus 1} z_k + \frac{1}{y_k \oplus 1} \right)^{-b_{kj}} & \text{otherwise} \end{cases}
$$

- Conformal case — $y_i \equiv 1$:

Transformation of FIs exactly reproduces CA

- General case:

The transformation rules are the u^0 term in the expression.
Can be extracted taking $u \to \infty$ limit.

$\to$ Cluster algebra encodes the transformations for every possible choice of ranks.

The superpotential

We assumed that whenever Seiberg-like duality generates a 2-cycle, this is removed by suitable *quadratic* superpotential term W_{quad}.

Highly non-trivial!

Q: given a quiver, there $\exists$ an R-symmetric superpotential W such that $\forall$ sequences of mutations, 2-cycles are always "massive"?

The superpotential

We assumed that whenever Seiberg-like duality generates a 2-cycle, this is removed by suitable *quadratic* superpotential term W_{quad}.

Highly non-trivial!

Q: given a quiver, there $\exists$ an R-symmetric superpotential W such that $\forall$ sequences of mutations, 2-cycles are always "massive"?

Such W: non-degenerate graded potential

Theory: non-degenerate if it admits such a potential.

- It is easy to produce examples of degenerate theories
- Complete classification of non-degenerate theories is *not* known
- E.g.: quivers dual to ideal triangulations of marked Riemann surfaces are non-degenerate.

The $\mathcal{Q}$-polynomial

Twisted chiral ring of the quiver: $Q_j(x) = \det(x - \sigma_j)$

$$\prod_i Q_i(x)^{[b_{ji}]_+} + i^{N_a(j)-N_f(j)} z_j \prod_i Q_i(x)^{[-b_{ji}]_+} = C_j(z_j)\, Q_j(x)\, T_j(x) \qquad \forall j$$

Under duality, identify $Q'_k(x) = T_k(x)$.

The $\mathcal{Q}$-polynomial

Twisted chiral ring of the quiver: $Q_j(x) = \det(x - \sigma_j)$

$$\prod_i Q_i(x)^{[b_{ji}]_+} + i^{N_a(j) - N_f(j)} z_j \prod_i Q_i(x)^{[-b_{ji}]_+} = C_j(z_j)\, Q_j(x)\, T_j(x) \qquad \forall j$$

Under duality, identify $Q'_k(x) = T_k(x)$.

- Define "dressed $\mathcal{Q}$-polynomials":

$$\mathcal{Q}_j(x) \simeq x_j \det(x - \sigma_j)$$

$\mathcal{Q}$-polynomials transform as cluster variables!

$$\mathcal{Q}'_j(x) = \begin{cases} \dfrac{\prod_i \mathcal{Q}_i(x)^{[b_{ki}]_+} + \prod_i \mathcal{Q}_i(x)^{[-b_{ki}]_+}}{\mathcal{Q}_k(x)} & j = k \\ \mathcal{Q}_j(x) & \text{otherwise} \end{cases}$$

Some applications

The quantum Kähler moduli space

"Conformal" quivers flow to IR CFTs: NLSM on (non-compact) Calabi-Yau's

Complexified FI parameters z_i control Kähler moduli

Metric on Kähler moduli space computed by [Jockers, Kumar, Lapan, Morrison, Romo 12]

$$Z_{S^2}(t_i, \bar{t}_i) = e^{-K_{\text{Kähler}}(t_i, \bar{t}_i)}$$

Quantum Kähler moduli space has cluster algebra structure

Also has Poisson structure.

The quantum Kähler moduli space

"Conformal" quivers flow to IR CFTs: NLSM on (non-compact) Calabi-Yau's

Complexified FI parameters z_i control Kähler moduli

Metric on Kähler moduli space computed by [Jockers, Kumar, Lapan, Morrison, Romo 12]

$$Z_{S^2}(t_i, \bar{t}_i) = e^{-K_{\text{Kähler}}(t_i, \bar{t}_i)}$$

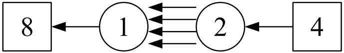
Quantum Kähler moduli space has cluster algebra structure

Also has Poisson structure.

- Compact example: Gulliksen-Negård CY_3

$$\mathcal{X} = \{\phi \in \mathbb{P}^7 \mid \operatorname{rank}(A^a_{4\times 4}\phi_a) \leq 2\} \qquad (h^{1,1}, h^{2,1}) = (2, 34)$$

Can be realized by a $U(1) \times U(2)$ quiver [Jockers, Kumar, Lapan, Morrison, Romo 12]



There is a cubic superpotential that breaks the flavor symmetry.

Quantum integrable systems (spin chains)

Nekrasov-Shatashvili: $\mathcal{N} = (2,2)$ gauge theory $\leftrightarrow$ quantum integrable system

Q: Integrable systems for our quivers? Seiberg-like dualities?

Quantum integrable systems (spin chains)

Nekrasov-Shatashvili: $\mathcal{N}=(2,2)$ gauge theory $\quad\leftrightarrow\quad$ quantum integrable system

Q: Integrable systems for our quivers? Seiberg-like dualities?

- $\mathcal{N}=(2,2)^*$ SQCD: $\begin{cases} U(N),\ N_f \text{ hypers (fund + antifund), one adjoint} \\ W=\tilde{Q}\Phi Q \end{cases}$

Param: cplx FI $t=2\pi\xi+i\theta$, twisted masses m_Φ, m_Q, $\tilde{m}_Q=-m_\Phi-m_Q$

- $SU(2)$ periodic twisted inhomogeneous $\mathrm{XXX}_{\frac{1}{2}}$ spin chain, N_f nodes, sector with $S_z=-\frac{N_f}{2}+N$ (N-particle states), inhomogeneous def's ν_a

$$H=J\sum_{a=1}^{N_f}\vec{S}_a\cdot\vec{S}_{a+1} \qquad\qquad \vec{S}_{N_f+1}=e^{\frac{i}{2}\vartheta\sigma_3}\,\vec{S}_1\,e^{-\frac{i}{2}\vartheta\sigma_3}$$

Dictionary: $z=e^{-t}=e^{i\vartheta}$, $m_\Phi=-iu$, $m_Q=\nu_a u+\frac{i}{2}u$, $\tilde{m}_Q=\nu_a u+\frac{i}{2}u$

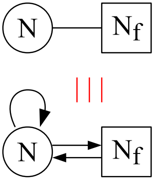
Twisted chiral ring relations $\quad\leftrightarrow\quad$ algebraic Bethe ansatz equations

Quantum integrable systems (spin chains) [in progress]

Strategy:

- Construct $\mathcal{N} = (2,2)^*$ "quivers"
- Look for a duality
- Limit of parameters $\quad \rightarrow \quad \mathcal{N} = (2,2)$ cycle-free quiver
- Interpret duality in integrable system context

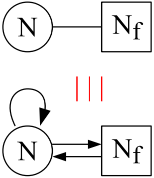
$\mathcal{N} = (2,2)^*$: diagrams with no arrows



Quantum integrable systems (spin chains) [in progress]

Strategy:
- Construct $\mathcal{N} = (2,2)^*$ "quivers"
- Look for a duality
- Limit of parameters $\rightarrow$ $\mathcal{N} = (2,2)$ cycle-free quiver
- Interpret duality in integrable system context

$\mathcal{N} = (2,2)^*$: diagrams with no arrows



"Particle-hole" duality: $U(N) \leftrightarrow U(N_f - N)$

$T^*\mathsf{Gr}(N, N_f) = T^*\mathsf{Gr}(N_f - N, N_f)$

- Limit: $m_\Phi, \tilde{m}_Q \to \infty$ with m_Q fixed.

It is crucial to expand in the correct vacuum.

Duality involves non-trival map of vacua.

Limit does not reproduce Seiberg-like duality directly, but its compatible with it.

Integrable system: "highly quantum and inhomogeneous limit" $(u, \nu_a \to \infty)$.

Conclusions

Open directions

- Cluster algebra in systems of Picard-Fuchs equations?

  GKZ-systems, A-systems, ... well-understood only for Abelian theories

- B-side of the story and Hori-Vafa mirror symmetry

- Relation of 2d $\mathcal{N} = (2, 2)$ quivers to other physical systems?

  Teichmüller theory and class $\mathcal{S}$ theories (what is Z_{S^2}? what is $K_{\text{Kähler}}$?)

  Wall crossing of BPS states in 4d $\mathcal{N} = 2$,

  DT invariants and quiver quantum mechanics

Thank you!