Mathieu Moonshine and Heterotic String Compactifications

Timm Wrase

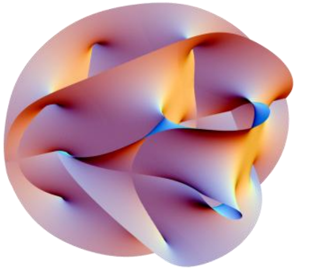
Texas A&M April 30, 2014

Based on: M. Cheng, X. Dong, J. Duncan, S. Harrison, S. Kachru, TW to appear
N. Paquette, TW to appear
TW 1402.2973
M. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, TW 1306.4981

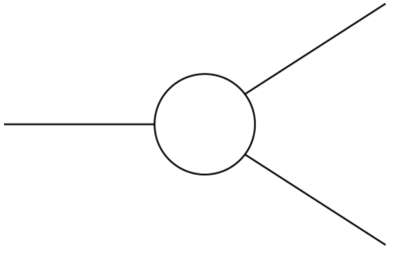
Outline

- Introduction to moonshine

- Mathieu Moonshine and string compactifications



- Physical implications



- A new moonshine phenomena

Finite simple groups

There are 18 infinite families, e.g.

- Alternating group of n elements A_n

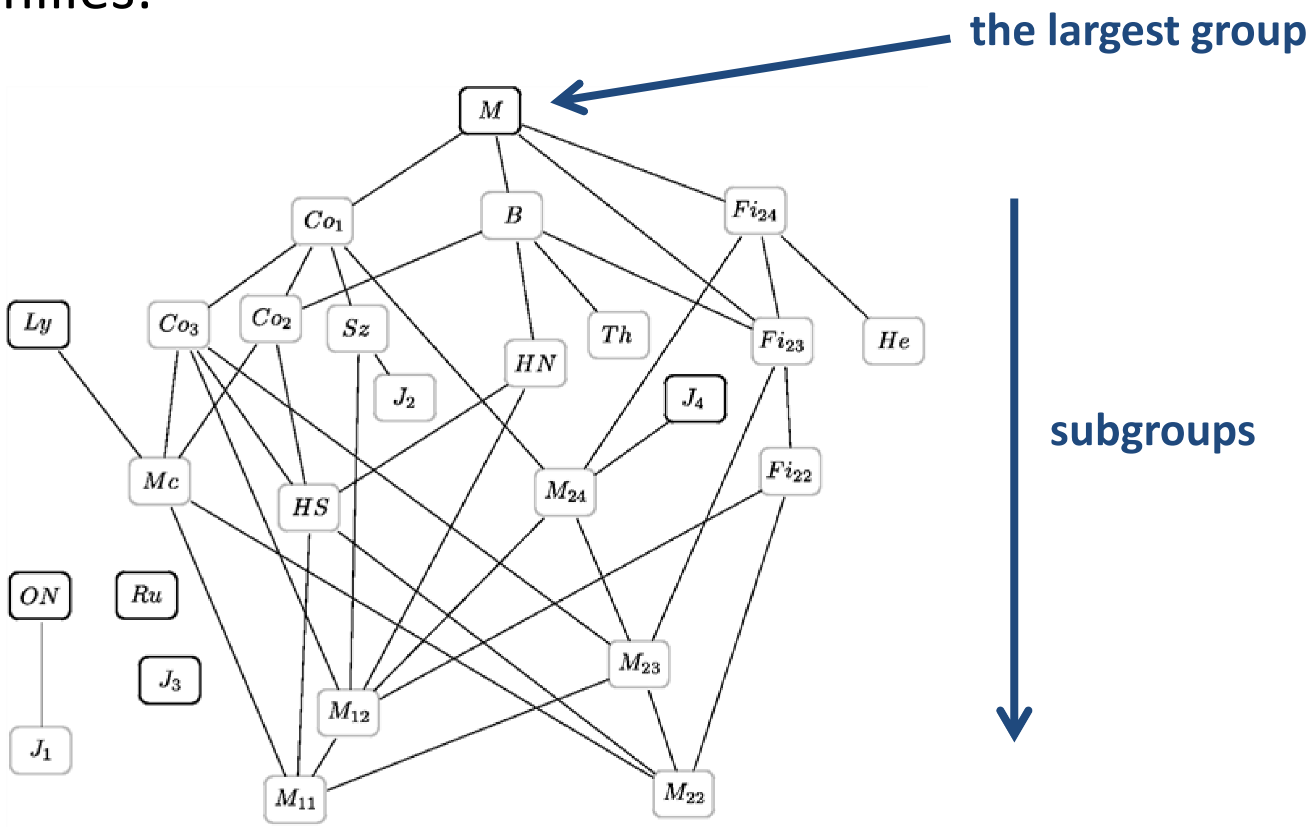
  e.g. A_3: (123) $\leftrightarrow$ (231) $\leftrightarrow$ (312)

- Cyclic groups of prime order C_p

  e.g.: $C_p = \mathbb{Z}_p = \langle e^{2\pi i/p} \rangle$

Finite simple groups

There are also 26 so called sporadic groups that do not come in infinite families:



Finite simple groups

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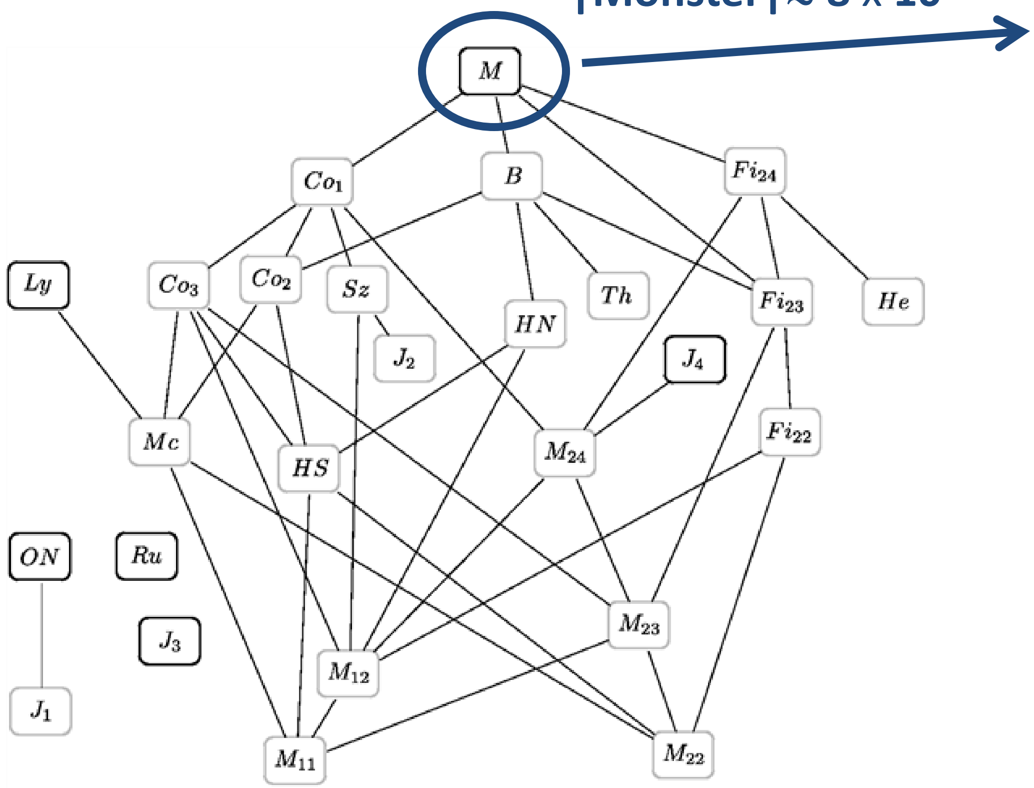
|Monster| ≈ 8 x 10^{53}



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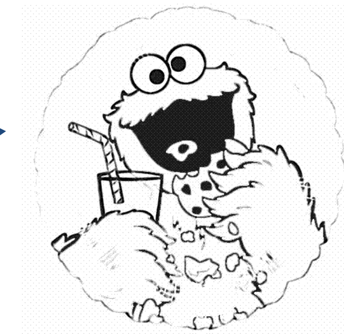
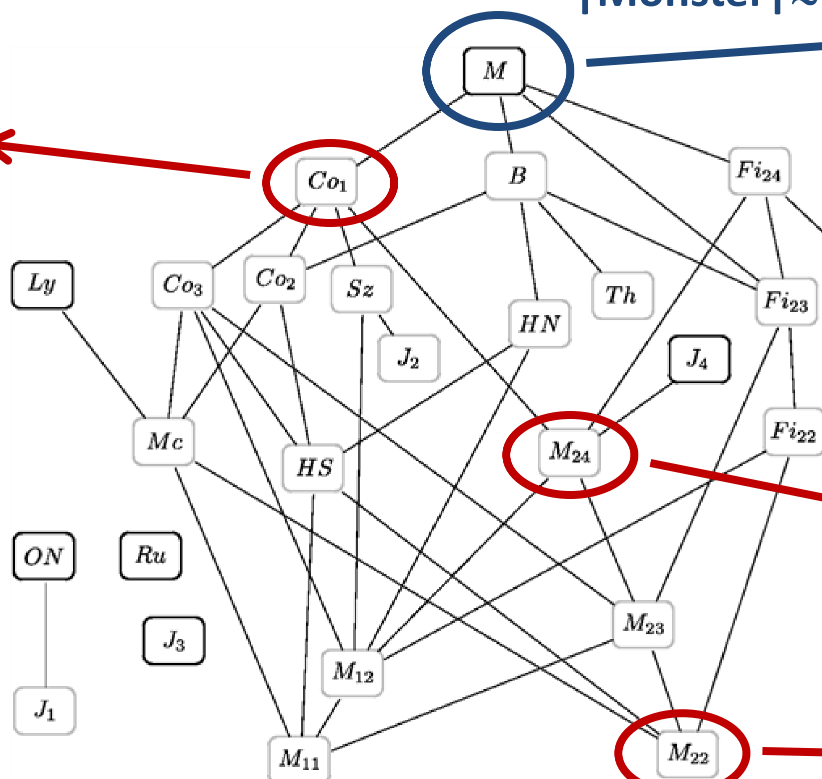


Conway group ≈ 4 x 10^{18}



Largest Mathieu group ≈ 2 x 10^9

$|M_{22}|$ = 443, 520

Modular Forms

Modular function of weight *k*

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k f(\tau), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

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Jacobi form of weight *k* and index *m*

$$f\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)^k e^{\frac{2\pi i m c z^2}{c\tau+d}} f(\tau,z)$$

$$f(\tau,z+\lambda\tau+\mu)=e^{-2\pi i m(\lambda^2\tau+\lambda z)} f(\tau,z), \qquad \lambda,\mu\in\mathbb{Z}$$

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$$f(\tau, z+\lambda\tau+\mu)=e^{-2\pi i m(\lambda^2\tau+\lambda z)} f(\tau,z), \qquad \lambda,\mu\in\mathbb{Z}$$

Can Fourier expand

$$f(\tau,z)= f(q=\exp[2\pi i\tau], y=\exp[2\pi i z])=\sum_{n\geq 0}\sum_{r^2\leq 4mn} c(n,r)\, q^n y^r$$

Monstrous Moonshine

- The irreducible representations of the Monster group have dimensions 1, 196 883, 21 296 876, …
- The J-function, that appears in many places in string theory, enjoys the expansion

$$J(q) = \frac{1}{q} + 196884\,q + 21493760\,q^2 + \ldots$$

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$$J(q) = \frac{1}{q} + \underbrace{196884}_{1 + 196883}\, q + \underbrace{21493760}_{1 + 196883 + 21296876}\, q^2 + \dots$$

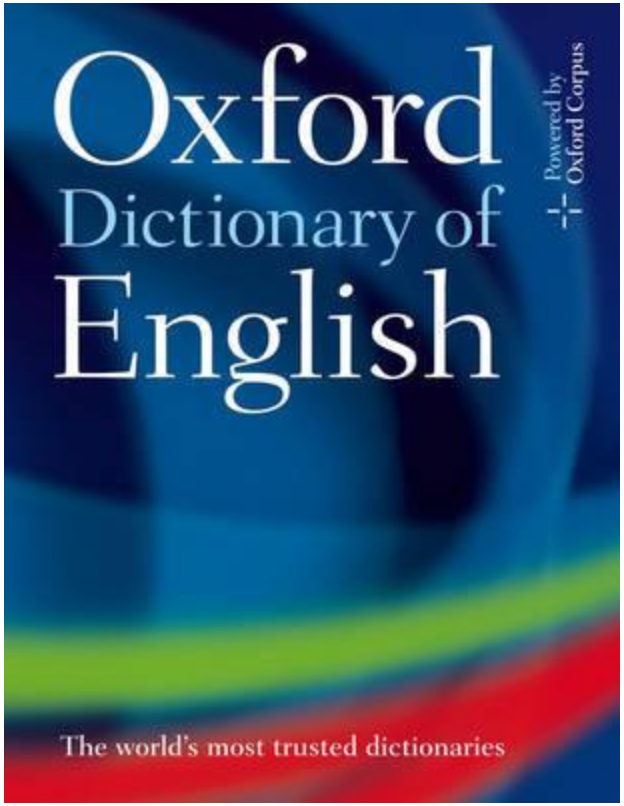
- as observed by John McKay

Monstrous Moonshine

moon·shine
ˈmo͞onˌSHīn/

noun informal
noun: **moonshine**

1. foolish talk or ideas.



Monstrous Moonshine

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noun informal

noun: **moonshine**

1. foolish talk or ideas.

2. NORTH AMERICAN
illicitly distilled or smuggled liquor.

Monstrous Moonshine

This surprising connection can be explained by string theory:

- The (left-moving) bosonic string compactified on a $\mathbb{Z}_2$ orbifold of $\mathbb{R}^{24}/\Lambda$ with Λ the Leech lattice has as its 1-loop partition function the *J(q)*-function

$$Z(q) = \mathrm{Tr}_{\mathrm{H}}\, q^{L_0 - \frac{c}{24}} = J(q) = \frac{1}{q} + 196884\, q + 21493760\, q^2 + \ldots$$

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no massless q^0 states

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tachyon of the bosonic string

supermassive string states

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- The symmetry group of the compactification space $\mathbb{R}^{24}/\Lambda/\mathbb{Z}_2$ is the Monster group.

Monstrous Moonshine

Since we have a Virasoro algebra we can expand the *J(q)*-function in terms of Virasoro characters (traces of Verma modules)

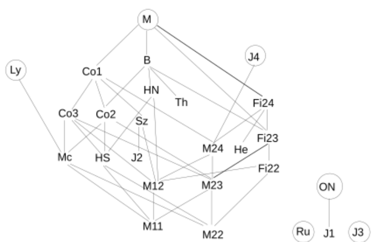
$$\mathrm{ch}_{\mathrm{h}=0}(q) = \frac{q^{-c/24}}{\prod_{n=2}^{\infty}(1-q^n)}\,, \qquad \mathrm{ch}_h(q) = \frac{q^{h-c/24}}{\prod_{n=1}^{\infty}(1-q^n)}$$

$$J(q) = \frac{1}{q} + 196884\,q + 21493760\,q^2 + \ldots$$

$$= \boxed{1}\,\mathrm{ch}_0(q) + \boxed{196883}\,\mathrm{ch}_2(q) + \boxed{21296876}\,\mathrm{ch}_3(q) + \ldots$$

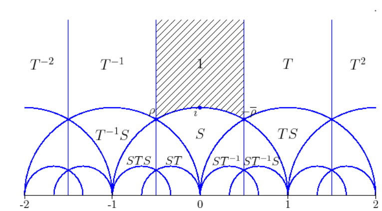
Monstrous Moonshine

Group Theory
Representation Theory
finite (sporadic) groups



String Theory

Complex Analysis
Number Theory
(mock) modular forms,
Jacobi forms



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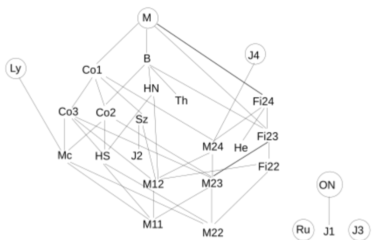

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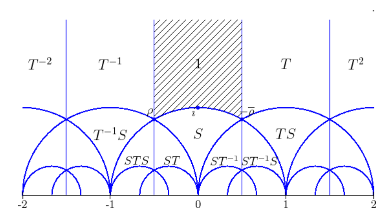

Very interesting for mathematicians!

Monstrous Moonshine

Compactification of the bosonic string:

$\Longrightarrow$ we have a tachyon (instability)

$\Longrightarrow$ spacetime theory has no fermions

Additionally

- Only two spacetime dimensions are non-compact

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Compactification of the bosonic string:

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Additionally

- Only two spacetime dimensions are non-compact

Not so interesting for physicists!

Mathieu Moonshine

- In 2010 Eguchi, Ooguri and Tachikawa discovered a new moonshine phenomenon that connects K3 to the largest Mathieu group M_{24}

Eguchi, Ooguri, Tachikawa 1004.0956

- They considered a N=(4,4) SCFT with K3 target and calculate an index that is called elliptic genus

Mathieu Moonshine

Chemical potential for U(1)
in left-moving N=2 theory

$$Z_{\text{elliptic}}(q,y) = \mathrm{Tr}_{\mathrm{RR}}\left((-1)^{F_L}\, q^{L_0 - \frac{c}{24}} y^{J_0} \underbrace{(-1)^{F_R}\, \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}} \right)$$

Witten index:
No dependence on $\bar{q}$

Mathieu Moonshine

$$Z^{\mathrm{K3}}_{\mathrm{elliptic}}(q,y)=8\left(\frac{\theta_2(q,y)^2}{\theta_2(q,1)^2}+\frac{\theta_3(q,y)^2}{\theta_3(q,1)^2}+\frac{\theta_4(q,y)^2}{\theta_4(q,1)^2}\right)$$

T. Eguchi, H. Ooguri, A. Taormina, S. -K. Yang Nucl. Phys. B 315, 193 (1989)

We have N=(4,4) world sheet supersymmetry

$\Longrightarrow$ expand in N=4 Virasoro characters

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N=4 Virasoro characters are defined by the trace over the highest weight state and all its descendants

$$\mathrm{ch}_{\mathrm{h,l}}(q,y)=\mathrm{Tr}\left(q^{L_0-\frac{c}{24}}y^{J_0}\right)$$

For the case h=c/24 there are short BPS multiplets

Mathieu Moonshine

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T. Eguchi, K. Hikami   0904.0911

$$Z^{\mathrm{K3}}_{\mathrm{elliptic}}=24\,\mathrm{ch}^{\mathrm{short}}_{h=\frac{1}{4},l=0}-2\,\mathrm{ch}^{\mathrm{long}}_{h=\frac{1}{4},l=\frac{1}{2}}+\sum_{n=1}^{\infty}A_n\,\mathrm{ch}^{\mathrm{long}}_{h=\frac{1}{4}+n,l=\frac{1}{2}}$$

$$A_n=\{90,\ 462,1440,\ldots\}$$

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T. Eguchi, H. Ooguri, Y. Tachikawa 1004.0956

$$A_n=\{45+\overline{45},\ 231+\overline{231},770+\overline{770},\ldots\}$$

Dimensions of Irreps of M_{24}

Mathieu Moonshine

Does this imply a connection between M_{24} and K3?

- The geometric symmetries of K3 are contained in $M_{23} \subset M_{24}$

Mukai, Kondo 1988, 1998

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Gaberdiel, Hohenegger, Volpato 1106.4315

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  Gaberdiel, Hohenegger, Volpato 1106.4315

- However, all coefficients $Z^{\mathrm{K3}}_{\mathrm{elliptic}}$ are positive sums of dimensions of M_{24}

  Gannon 1211.5531

Mathieu Moonshine

- K3 has played a central role in string compactifications and string dualities
- What are implications we can derive from Mathieu moonshine for string compactifications?
- Has the elliptic genus of K3 already appeared in the string theory literature?

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- What are implications we can derive from Mathieu moonshine for string compactifications?
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YES!

Heterotic String Theory

- Consider the heterotic E_8 x E_8 string theory compactified on K3 x T^2

- We need to embed 24 instantons into E_8 x E_8 $\longrightarrow$ (12+n,12-n) n = 0,1,…,12 to satisfy the Bianchi identity for H_3

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- We need to embed 24 instantons into E_8 x E_8 $\longrightarrow$ (12+n,12-n) n = 0,1,…,12 to satisfy the Bianchi identity for H_3
- The resulting four dimensional theories preserves N=2 spacetime supersymmetry
- The 1-loop corrections to the prepotential are related to the new supersymmetric index Z_{new}

Dixon, Kaplunovsky, Louis, de Wit, Lüst, Stieberger, Antoniadis, Narain, Taylor, Gava, Kiritsis, Kounnas, Harvey, Moore, ….

$$h(S,T,U) = h^{\text{tree}} + h^{1-loop} + O(e^{-2\pi i S})$$

Heterotic String Theory

- The new supersymmetric index is defined as

$$Z_{\text{new}} = \text{Tr}_{\text{R}}\left(\bar{J}_0 (-1)^{\bar{J}_0} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}\right)$$

- The trace is over our internal (22,9) conformal field theory for the heterotic $E_8 x E_8$ string theory compactified on K3 x T^2

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- The trace is over our internal (22,9) conformal field theory for the heterotic $E_8 x E_8$ string theory compactified on K3 x T^2
- We have a right moving N=2 SCFT from the T^2 and we denote its U(1) generator $\bar{J}^{(1)}$
- For the K3 we have an N=4 SCFT with a level one SU(2). We define $\bar{J}^{(2)} = 2\,\bar{J}^3$ where $\bar{J}^3$ is the SU(2) Cartan current
- Then $\bar{J} = \bar{J}^{(1)} + \bar{J}^{(2)}$

Heterotic String Theory

- The new supersymmetric index is defined as

$$Z_{\text{new}} = \text{Tr}_{\text{R}} \left((\bar{J}_0^{(1)} + \bar{J}_0^{(2)})(-1)^{\bar{J}_0^{(1)} + \bar{J}_0^{(2)}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right)$$

$$= \text{Tr}_{\text{R}} \bar{J}_0^{(1)} (-1)^{\bar{J}_0^{(1)} + \bar{J}_0^{(2)}} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}$$

$$+ \text{Tr}_{\text{R}} (-1)^{\bar{J}_0^{(1)}} q^{L_0 - \frac{c}{24}} \bar{J}_0^{(2)} (-1)^{\bar{J}_0^{(2)}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}$$

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$$+ \mathrm{Tr}_{\mathrm{R}} (-1)^{\bar{J}_0^{(1)}} q^{L_0-\frac{c}{24}} \bar{J}_0^{(2)} (-1)^{\bar{J}_0^{(2)}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}$$

For SU(2) representations eigenvalues of $J^{(2)}$ come in opposite pairs

$$\mathrm{Tr}_{\mathrm{K3}} \bar{J}_0^{(2)} (-1)^{\bar{J}_0^{(2)}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}$$

$$\approx \sum_{n\in\mathbb{Z}} (n(-1)^n - n(-1)^{-n})[\ldots] = 0$$

Heterotic String Theory

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0

Heterotic String Theory

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$$
\begin{aligned}
Z_{\text{new}} &= \mathrm{Tr}_{\mathrm{R}}\left((\bar{J}_0^{(1)} + \bar{J}_0^{(2)})(-1)^{\bar{J}_0^{(1)}+\bar{J}_0^{(2)}} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} \right) \\
&= \mathrm{Tr}_{\mathrm{R}} \bar{J}_0^{(1)} (-1)^{\bar{J}_0^{(1)}+\bar{J}_0^{(2)}} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}} \\
&\quad + \mathrm{Tr}_{\mathrm{R}} (-1)^{\bar{J}_0^{(1)}} q^{L_0-\frac{c}{24}} \underbrace{\bar{J}_0^{(2)} (-1)^{\bar{J}_0^{(2)}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}}_{\to\, 0} \\
&= \mathrm{Tr}_{\mathrm{R}} q^{L_0-\frac{c}{24}} \bar{J}_0^{(1)} (-1)^{\bar{J}_0^{(1)}+\bar{J}_0^{(2)}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}
\end{aligned}
$$

- Now we calculate it for the standard embedding $SU(2) \subset E_8$ for a compactification on K3 x T^2

Heterotic String Theory

For K3 x T^2 compactifications we have for the standard embedding that preserves N=(4,4)

Witten index

$$Z_{\text{new}} = \mathrm{Tr}_{\mathrm{R}}\, q^{L_0 - \frac{c}{24}} \underbrace{\bar{J}_0^{(1)} (-1)^{\bar{J}_0^{(1)} + \bar{J}_0^{(2)}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}}$$

Compare to

$$Z_{\text{elliptic}}(q,y) = \mathrm{Tr}_{\mathrm{RR}} \left((-1)^{F_L} q^{L_0 - \frac{c}{24}} y^{J_0} (-1)^{F_R} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}} \right)$$

We will get contributions from $Z^{\mathrm{K3}}_{\text{elliptic}}(q,y)$ at different *y*-values

Heterotic String Theory

For K3 x T^2 compactifications we have for the standard embedding that preserves N=(4,4)

Affine E_8

SO(12) characters

$$Z_{\text{new}}(q;T,U)=\frac{i}{2}\frac{\Theta_{\Gamma_{2,2}}(T,U)}{\eta(q)^4}\frac{\mathrm{E}_4(\mathrm{q})}{\eta(\mathrm{q})^8}\Bigg[\left(\frac{\theta_2(q)}{\eta(q)}\right)^6 Z^{\text{K3}}_{\text{elliptic}}(q,-1)+\left(\frac{\theta_3(q)}{\eta(q)}\right)^6 q^{\frac{1}{4}} Z^{\text{K3}}_{\text{elliptic}}(q,-q^{\frac{1}{2}})-\left(\frac{\theta_4(q)}{\eta(q)}\right)^6 q^{\frac{1}{4}} Z^{\text{K3}}_{\text{elliptic}}(q,q^{\frac{1}{2}})\Bigg]$$

SO(12) characters

Harvey, Moore hep-th/9510182

T is the complexified Kähler modulus, *U* the complex structure modulus of the T^2

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SO(12) characters

Harvey, Moore hep-th/9510182

So in particular the […] part has an "SO(12)xM_{24}"-expansion: exactly the same M_{24} as in Mathieu Moonshine due to *N=(4,4)*

Heterotic String Theory

For K3 x T^2 compactifications we have for the standard embedding that preserves N=(4,4)

$$Z_{\rm new}(q;T,U)=\frac{i}{2}\frac{\Theta_{\Gamma_{2,2}}(T,U)}{\eta(q)^4}\frac{\mathrm{E}_4(\mathrm{q})}{\eta(\mathrm{q})^8}\Bigg[$$

$$+24\left(\left(\frac{\theta_2(q)}{\eta(q)}\right)^6 \mathrm{ch}^{\rm short}_{h=\frac{1}{4},l=0}(q,-1)+\left(\frac{\theta_3(q)}{\eta(q)}\right)^6 q^{\frac{1}{4}}\mathrm{ch}^{\rm short}_{h=\frac{1}{4},l=0}(q,-q^{\frac{1}{2}})-\left(\frac{\theta_4(q)}{\eta(q)}\right)^6 q^{\frac{1}{4}}\mathrm{ch}^{\rm short}_{h=\frac{1}{4},l=0}(q,q^{\frac{1}{2}})\right)$$

$$-2\left(\left(\frac{\theta_2(q)}{\eta(q)}\right)^6 \mathrm{ch}^{\rm short}_{h=\frac{1}{4},l=\frac{1}{2}}(q,-1)+\left(\frac{\theta_3(q)}{\eta(q)}\right)^6 q^{\frac{1}{4}}\mathrm{ch}^{\rm short}_{h=\frac{1}{4},l=\frac{1}{2}}(q,-q^{\frac{1}{2}})-\left(\frac{\theta_4(q)}{\eta(q)}\right)^6 q^{\frac{1}{4}}\mathrm{ch}^{\rm short}_{h=\frac{1}{4},l=\frac{1}{2}}(q,q^{\frac{1}{2}})\right)$$

$$+\left(\left(\frac{\theta_2(q)}{\eta(q)}\right)^6 \mathrm{ch}^{\rm long}_{h=\frac{1}{4},l=\frac{1}{2}}(q,-1)+\left(\frac{\theta_3(q)}{\eta(q)}\right)^6 q^{\frac{1}{4}}\mathrm{ch}^{\rm long}_{h=\frac{1}{4},l=\frac{1}{2}}(q,-q^{\frac{1}{2}})-\left(\frac{\theta_4(q)}{\eta(q)}\right)^6 q^{\frac{1}{4}}\mathrm{ch}^{\rm long}_{h=\frac{1}{4},l=\frac{1}{2}}(q,q^{\frac{1}{2}})\right)\sum_{n=1}^{\infty}A_n\,q^n\Bigg]$$

$$A_n=\{45+\overline{45},\ 231+\overline{231},770+\overline{770},\ldots\}$$

Dimensions of Irreps of M_{24}

Heterotic String Theory

For K3 x T^2 compactifications we have for the standard (24,0) instanton embedding

Affine E_8

↓

$$Z_{\mathrm{new}}(q;T,U) = \frac{i}{2}\frac{\Theta_{\Gamma_{2,2}}(T,U)}{\eta(q)^4}\frac{\mathrm{E}_4(\mathrm{q})}{\eta(\mathrm{q})^8}\frac{\mathrm{E}_6(\mathrm{q})}{\eta(\mathrm{q})^{12}} = \frac{i}{2}\frac{\Theta_{\Gamma_{2,2}}(T,U)\mathrm{E}_4(\mathrm{q})\mathrm{E}_6(\mathrm{q})}{\eta(\mathrm{q})^{24}}$$

Harvey, Moore hep-th/9510182

So in particular the E_6(q) has an “SO(12)xM_{24}”-expansion

Heterotic String Theory

For K3 x T^2 compactifications we have for the standard embedding that preserves N=(4,4)

Take away message:

Z_{new} depends on T and U and is connected

to $Z_{elliptic}$ and therefore to M_{24}

T is the complexified Kähler modulus, *U* the complex structure modulus of the T^2

Heterotic String Theory

The 1-loop correction to the prepotential is roughly determined by

$$\Delta(T,U) = \int \frac{d^2\tau}{\tau_2} Z_{\text{new}}(q = e^{2\pi i\tau}; T,U)\,(Q^2 - \tfrac{1}{8\pi\tau_2})$$

and knows about M_{24} since Z_{new} does

M. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, TW 1306.4981

Heterotic String Theory

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M. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, TW 1306.4981

The modular invariance of $\tau_2 Z_{\text{new}}(Q^2 - \tfrac{1}{8\pi\tau_2})$ actually tells us that there is a unique solution. So for all instanton embeddings (12+n,12-n) the answer is the same.

Kiritsis, Kounnas, Petropoulos, Rizos hep-th/9608034

Henningson, Moore hep-th/9608145

Heterotic String Theory

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M. Cheng, X. Dong, J. Duncan, J. Harvey, S. Kachru, TW 1306.4981

We have to solve the following second order differential equation

Harvey, Moore hep-th/9510182

$$-\mathrm{Re}\left(\partial_T \partial_U h^{1-loop} + \frac{1}{T_1 U_1}(1 - T_1\partial_T - U_1 \partial_U) h^{1-loop}\right) - \frac{1}{\pi}\mathrm{Re}(\log[J(iT) - J(iU)])$$

$$= \frac{1}{2\pi}\int \frac{d^2\tau}{\tau_2}\left(-iZ_{new}(q;T,U)\cdot(Q_{E_8}^2 - \tfrac{1}{8\pi\tau_2}) - b(E_8)\right) + \frac{b(E_8)}{2\pi}(\log[2T_1U_1] + 4\,\mathrm{Re}(\log[\eta(iT)\eta(iU)]))$$

Heterotic String Theory

The solution is given by

$$h^{1-loop} = -\tfrac{1}{3}U^3 + C + \sum_{k,l} c(kl) Li_3(e^{2\pi i(kT+lU)})$$

where the polylogarithm is given by $Li_3(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3}$

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and the expansion coefficients are the same as in our index (they go along for the ride when integrating)

$$Z_{\text{new}}(q;T,U) = \frac{i}{2}\Theta_{\Gamma_{2,2}}(T,U)\left(\sum_{m\geq -1} c(m)\, q^m\right)$$

Dimensions of M_{24}

Heterotic String Theory

The solution is given by

Dimensions of M_{24}
(appearing in a spacetime quantity)

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Dimensions of M_{24}

Type IIA on CY_3

String duality

Heterotic string
on K3 x T^2 with
instanton embedding
(12+n,12-n)

Type IIA string theory
on elliptic fibrations
over F_n (Hirzebruch
surface)

Type IIA on CY_3

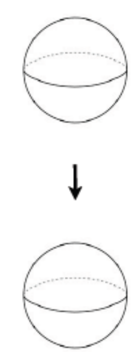
String duality

Heterotic string on $K3 \times T^2$ with instanton embedding (12+n,12-n)

Type IIA string theory on elliptic fibrations over F_n (Hirzebruch surface)

F_n

dilaton S ⟷ Size of base S^2



Type IIA on CY_3

Type IIA string theory on elliptic fibrations over F_n:

- Prepotential receives instanton corrections
- These are determined by the Gromow-Witten invariants ≈ curve counting (S^2, T^2, …)

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M. Cheng, X. Dong, J. Duncan,
J. Harvey, S. Kachru, TW 1306.4981

$$h(S,T,U) = -STU - \tfrac{1}{3}U^3 + C + \sum_{k,l} c(kl) Li_3(e^{2\pi i(kT+lU)}) + O(e^{-2\pi i S})$$

M. Alim, E. Scheidegger 1205.1784
A. Klemm, J. Manschot, T. Wotschke 1205.1795

Dimensions of M_{24}
(appearing in a spacetime quantity)

Type IIB on CY_3

String duality

Type IIA string theory on elliptic fibrations over F_n (Hirzebruch surface)

Type IIB string theory on mirror manifold

Type IIB on CY_3

String duality

Type IIA string theory on elliptic fibrations over F_n (Hirzebruch surface)	Type IIB string theory on mirror manifold

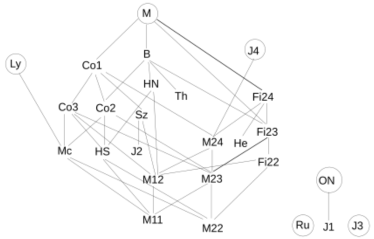
CY_3 manifold X_n ⟷ CY_3 manifold Y_n

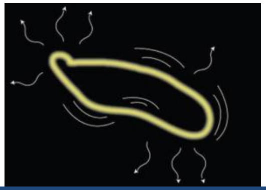
Gromov-Witten invariants ⟷ Periods of the holo-morphic 3-form Ω

Cool new math connections!

Group Theory
Representation Theory

finite (sporadic) groups



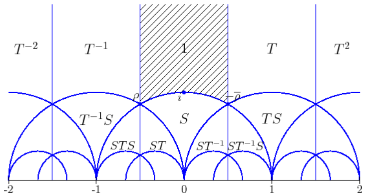


String Theory

NEW

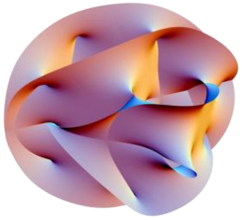
Complex Analysis
Number Theory

(mock) modular forms,
Jacobi forms



(Algebraic) Geometry

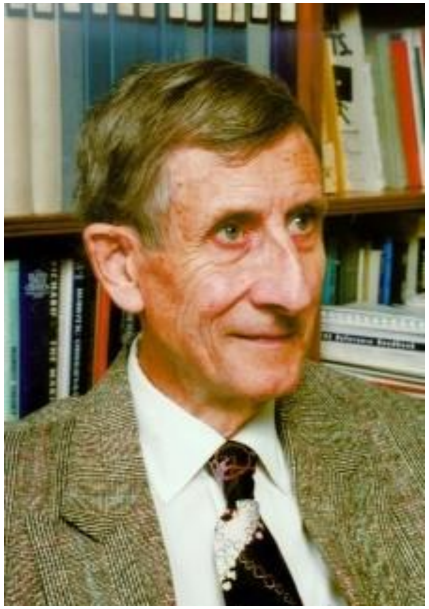
periods of Calabi-Yau manifolds,
Gromov-Witten
invariants,
elliptic genus,
…



Moonshine and physics

"I have a sneaking hope, a hope unsupported by any facts or any evidence, that sometime in the twenty-first century physicists will stumble upon the Monster group, built in some unsuspected way into the structure of the Universe."

– Freeman Dyson (1983)



Moonshine and physics

For the K3xT^2 compactifications, the 1-loop prepotential controls the 1-loop corrections to the gauge couplings in the N=2 spacetime theory

Moonshine and physics

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For four dimensional N=1 models obtained from orbifold compactifications of the heterotic E_8 x E_8 string theory:

$$f_\alpha(S,T,U) = S + f_\alpha^{1\text{-loop}}(T,U) + O(e^{-2\pi i S})$$

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$$f_\alpha(S,T,U) = S + f_\alpha^{1-\text{loop}}(T,U) + O(e^{-2\pi i S})$$

The (bulk) moduli dependent 1-loop correction to the gauge kinetic function arises only from N=2 subsectors!

Dixon, Louis, Kaplunovsky Nuclear Physics B 355 (1991)

Moonshine and physics

Example $T^6/\mathbb{Z}_{6\text{-}II}$ = $T^2 \times T^2 \times T^2/\mathbb{Z}_{6\text{-}II}$:

$$\mathbb{Z}_{6-II} = \langle g \rangle, \qquad g : (z_1, z_2, z_3) \to (e^{\pi i/3} z_1, e^{2\pi i/3} z_2, -z_3)$$

Moonshine and physics

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has two N=2 subsector

$$\mathbb{Z}_3 = \{1, g^2, g^4\}, \qquad g^2 : (z_1, z_2, z_3) \rightarrow (e^{2\pi i/3} z_1, e^{4\pi i/3} z_2, z_3)$$

$$\mathbb{Z}_2 = \{1, g^3\}, \qquad g^3 : (z_1, z_2, z_3) \rightarrow (-z_1, z_2, -z_3)$$

Moonshine and physics

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$$\mathbb{Z}_2 = \{1, g^3\}, \qquad g^3 : (z_1, z_2, z_3) \to (-z_1, z_2, -z_3)$$

For which the internal space is $T^4/\mathbb{Z}_3 \times T^2$ or $T^4/\mathbb{Z}_2 \times T^2$ respectively and therefore an orbifold limit of $T^2 \times K3$.

Moonshine and physics

N=2 sectors lead to 1-loop corrections

N=2 prepotential

$$f_{\alpha}^{1-\mathrm{loop}}(T,U)=\sum_{i=1,2,3}\frac{|G_i'|}{|G|}\Bigg[-\frac{1}{2}\partial_{T_i}\partial_{U_i}h_i^{1-loop}(T_i,U_i)$$

$$-\frac{1}{8\pi^2}\log[J(iT_i)-J(iU_i)-\frac{b_{\alpha,i}^{N=2}}{4\pi^2}(\log[\eta(iT_i)\eta(iU_i)])\Bigg]$$

N=1 gauge kinetic coupling

Moonshine and physics

N=2 sectors lead to 1-loop corrections

N=2 prepotential

$$f_\alpha^{1-\text{loop}}(T,U) = \sum_{i=1,2,3} \frac{|G_i'|}{|G|}\Bigg[-\frac{1}{2}\partial_{T_i}\partial_{U_i} h_i^{1-loop}(T_i,U_i)$$

N=1 gauge kinetic coupling

$$-\frac{1}{8\pi^2}\log[J(iT_i)-J(iU_i)-\frac{b_{\alpha,i}^{N=2}}{4\pi^2}(\log[\eta(iT_i)\eta(iU_i)]) \Bigg]$$

where the prepotential was calculated above

$$h^{1-loop}(T,U) = -\tfrac{1}{3}U^3 + C + \sum_{k,l} c(kl) Li_3(e^{2\pi i(kT+lU)})$$

Dimensions of M_{24}

Moonshine and physics

Group $\mathbb{Z}_N$	Generator $\frac{1}{N}(\varphi_1, \varphi_2, \varphi_3)$	$\mathcal{N} = 2$ moduli
$\mathbb{Z}_3$	$\frac{1}{3}(1,1,1)$	-
$\mathbb{Z}_4$	$\frac{1}{4}(1,1,2)$	T_3, U_3
$\mathbb{Z}_{6-I}$	$\frac{1}{6}(1,1,4)$	T_3
$\mathbb{Z}_{6-II}$	$\frac{1}{6}(1,2,3)$	T_2, T_3, U_3
$\mathbb{Z}_7$	$\frac{1}{7}(1,2,4)$	-
$\mathbb{Z}_{8-I}$	$\frac{1}{8}(1,2,5)$	T_2
$\mathbb{Z}_{8-II}$	$\frac{1}{8}(1,3,4)$	T_3, U_3
$\mathbb{Z}_{12-I}$	$\frac{1}{12}(1,4,7)$	T_2
$\mathbb{Z}_{12-II}$	$\frac{1}{12}(1,5,6)$	T_3, U_3

No N=2 sectors

Moonshine and physics

$\mathbb{Z}_N \times \mathbb{Z}_M$	1^{st} generator $\frac{1}{N}(\varphi_1, \varphi_2, \varphi_3)$	2^{nd} generator $\frac{1}{M}(\hat{\varphi}_1, \hat{\varphi}_2, \hat{\varphi}_3)$	$\mathcal{N} = 2$ moduli
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\frac{1}{2}(1,0,1)$	$\frac{1}{2}(0,1,1)$	$T_1, U_1, T_2, U_2, T_3, U_3$
$\mathbb{Z}_2 \times \mathbb{Z}_4$	$\frac{1}{2}(1,0,1)$	$\frac{1}{4}(0,1,3)$	T_1, U_1, T_2, T_3
$\mathbb{Z}_2 \times \mathbb{Z}_6$	$\frac{1}{2}(1,0,1)$	$\frac{1}{6}(0,1,5)$	T_1, U_1, T_2, T_3
$\mathbb{Z}_2 \times \mathbb{Z}_6'$	$\frac{1}{2}(1,0,1)$	$\frac{1}{6}(1,1,4)$	T_1, T_2, T_3
$\mathbb{Z}_3 \times \mathbb{Z}_3$	$\frac{1}{3}(1,0,2)$	$\frac{1}{3}(0,1,2)$	T_1, T_2, T_3
$\mathbb{Z}_3 \times \mathbb{Z}_6$	$\frac{1}{3}(1,0,2)$	$\frac{1}{6}(0,1,5)$	T_1, T_2, T_3
$\mathbb{Z}_4 \times \mathbb{Z}_4$	$\frac{1}{4}(1,0,3)$	$\frac{1}{4}(0,1,3)$	T_1, T_2, T_3
$\mathbb{Z}_6 \times \mathbb{Z}_6$	$\frac{1}{6}(1,0,5)$	$\frac{1}{6}(0,1,5)$	T_1, T_2, T_3

Moonshine and physics

Four dimensional N=1 models obtained from orbifold compactifications of the heterotic E_8 x E_8 string theory receive universal 1-loop corrections to their gauge kinetic functions that are related to M_{24}

TW 1402.2973

Moonshine and physics

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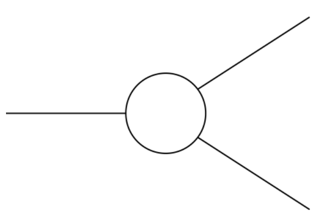
TW 1402.2973

For all $T^6/\mathbb{Z}_N$, N≠3,7, and all $T^6/\mathbb{Z}_N x \mathbb{Z}_M$

$$f(S,T,U) \approx S + \partial_T \partial_U \sum_{k,l} c(kl) Li_3(e^{2\pi i(kT+lU)}) + \ldots + O(e^{-2\pi i S})$$



Dimensions of M_{24}

Moonshine and physics

N. Paquette, TW work in progress:

- The holomorphic 3-form Ω plays a role in flux compactifications

Gukov, Vafa, Witten hep-th/9906070
Giddings, Kachru, Polchinski hep-th/0105097

$$W = \int H \wedge \Omega + \ldots$$

Moonshine and physics

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Flux vacua might have large symmetry groups: $|M_{24}| \approx 2 \times 10^9$

Moonshine and physics

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Gukov, Vafa, Witten hep-th/9906070
Giddings, Kachru, Polchinski hep-th/0105097

$$W = \int H \wedge \Omega + \dots$$

- The Yukawa couplings in heterotic models are given by the third derivative of Ω with respect to the moduli

Hosono, Klemm, Theisen, Yau hep-th/9308122

$$Y_{IJK} \approx \partial_I \partial_J \partial_K h(S,T,U)$$

New Moonshine phenomena

Consider eight (left-moving) bosons and fermions compactified on the orbifold $T^8/\mathbb{Z}_2 = \mathbb{R}^8/\Lambda_{E8}/\mathbb{Z}_2$

The partition function in the NS sector is

Frenkel, Lepowsky, Meurman 1985
Duncan math/0502267

$$Z(q) = \frac{1}{\sqrt{q}} + 276\sqrt{q} + 2048q + 11202q^{\frac{3}{2}} + \ldots$$

Sums of dimensions of Conway group

New Moonshine phenomena

Consider eight (left-moving) bosons and fermions compactified on the orbifold $T^8/\mathbb{Z}_2 = \mathbb{R}^8/\Lambda_{E8} / \mathbb{Z}_2$

Tensor left- and right movers together in an asymmetric orbifold where we act with one $\mathbb{Z}_2$ on the left and with another one only on the right:

$$Z^{\text{partition}}(q,\bar{q}) = Z(q)\,Z(\bar{q})$$

The asymmetric orbifold preserves N=(4,4) symmetry

New Moonshine phenomena

Asymmetric orbifold $T^8/\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{R}^8/\Lambda_{E8}/\mathbb{Z}_2 \times \mathbb{Z}_2$

Calculate the elliptic genus and expand in N=4 characters:

$$\begin{aligned} Z_{\text{elliptic}} &= 21\,\mathrm{ch}^{\text{short}}_{h=\frac{1}{2},l=0} + \mathrm{ch}^{\text{short}}_{h=\frac{1}{2},l=1} \\ &+ 560\,\mathrm{ch}^{\text{long}}_{h=\frac{3}{2},l=\frac{1}{2}} + 8470\,\mathrm{ch}^{\text{long}}_{h=\frac{5}{2},l=\frac{1}{2}} + \ldots \\ &+ 210\,\mathrm{ch}^{\text{long}}_{h=\frac{3}{2},l=1} + 4444\,\mathrm{ch}^{\text{long}}_{h=\frac{5}{2},l=1} + \ldots \end{aligned}$$

Two infinite series, coefficients unrelated to Conway

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$$+ 560\,\text{ch}^{\text{long}}_{h=\frac{3}{2},l=\frac{1}{2}} + 8470\,\text{ch}^{\text{long}}_{h=\frac{5}{2},l=\frac{1}{2}} + \ldots$$
$$+ 210\,\text{ch}^{\text{long}}_{h=\frac{3}{2},l=1} + 4444\,\text{ch}^{\text{long}}_{h=\frac{5}{2},l=1} + \ldots$$

All coefficients are dimensions of the Mathieu group M_{22}!

Checks all work and one can see $M_{22} \subset Co$ preserving N=4

M. Cheng, X. Dong, J. Duncan, S. Harrison, S. Kachru, TW to appear

Conclusion

- Mathieu Moonshine involves K3 that has played a crucial role in superstring compactifications and string dualities
- Certain CY_3 manifolds are now also implicated in Mathieu Moonshine
- This leads to a variety of intriguing implications for physically interesting string compactifications
- Much more to come!

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THANK YOU!