What Happens to the Schwarzschild Solution in Higher-Derivative Gravity?

K.S. Stelle

Imperial College London

Mitchell Institute Workshop
Texas A & M University
College Station Texas

May 1, 2015

K.S.S., Gen.Rel.Grav. 9 (1978) 353
H. Lü, Y. Pang, C.N. Pope & J.F. Vásquez-Poritz, arXiv:1204.1062
H. Lü, A. Perkins, C.N. Pope & K.S.S.,
PRL 114, 171601 (2015); arXiv 1502.01028
H. Lü, A. Perkins, C.N. Pope & K.S.S., in preparation

Quantum Context

One-loop quantum corrections to General relativity in 4-dimensional spacetime produce ultraviolet divergences of curvature-squared structure.

G. 't Hooft and M. Veltman, *Ann. Inst. Henri Poincaré* **20**, 69 (1974)

Inclusion of $\int d^4x\sqrt{-g}(\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2)$ terms ab initio in the gravitational action leads to a renormalizable $D = 4$ theory, but at the price of a loss of *unitarity* owing to the modes arising from the $\alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$ term, where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor.

K.S.S., *Phys. Rev.* **D16**, 953 (1977).

[In $D = 4$ spacetime dimensions, this (Weyl)^2 term is equivalent, up to a topological total derivative *via* the Gauss-Bonnet theorem, to the combination $\alpha(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2)$].

Despite the apparent nonphysical behavior, quadratic-curvature gravities continue to be explored in a number of contexts:

- *Cosmology:* Starobinsky's original model for inflation was based on a $\int d^4x\sqrt{-g}(-R+\beta R^2)$ model.

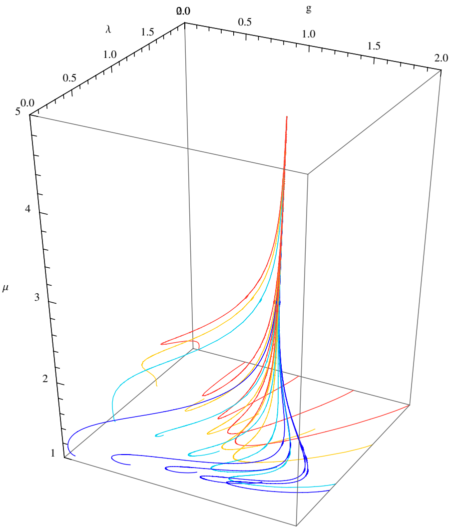
  A.A. Starobinsky 1980; Mukhanov & Chibisov 1981

  This early model has been quoted (at times) as a good fit to CMB fluctuation data from the Planck satellite.

  J. Martin, C. Ringeval and V. Vennin, arXiv:1303.3787

- The *asymptotic safety scenario* considers the possibility that there may be a non-Gaussian renormalization-group fixed point and associated flow trajectories on which the ghost states arising from the $(\mathrm{Weyl})^2$ term could be absent.

  S. Weinberg 1976, M. Reuter 1996, M. Niedermaier 2009



Renormalization-group trajectories in coupling-constant space ending on a non-Gaussian fixed point with finite g_{Newton} and cosmological constant Λ.

Niedermaier 2009

Classical gravity with higher derivatives

We shall not try here to settle philosophical debates about various attitudes that can be taken towards the implementation of quantum corrections, but shall simply adopt a point of view taking the higher-derivative terms and their consequences for gravitational field-theory solutions seriously in the classical effective action.

Accordingly, we shall consider the gravitational action

$$I = \int d^4x\sqrt{-g}(\gamma R - \alpha C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \beta R^2)\,.$$

The field equations following from this higher-derivative action are

$$\begin{aligned} H_{\mu\nu} &= \gamma\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \frac{2}{3}(\alpha - 3\beta)\nabla_\mu\nabla_\nu R - 2\alpha\Box R_{\mu\nu} \\ &+ \frac{1}{3}(\alpha + 6\beta)g_{\mu\nu}\Box R - 4\alpha R^{\eta\lambda}R_{\mu\eta\nu\lambda} + 2\left(\beta + \frac{2}{3}\alpha\right)RR_{\mu\nu} \\ &+ \frac{1}{2}g_{\mu\nu}\left(2\alpha R^{\eta\lambda}R_{\eta\lambda} - \left(\beta + \frac{2}{3}\alpha\right)R^2\right) = \frac{1}{2}T_{\mu\nu} \end{aligned}$$

Full nonlinear field equations for spherical symmetry

Use Schwarzschild coordinates

$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$

The first equation contains the third-order derivative $B^{(3)} = B'''$ (Written in an older parametrization. To obtain the new parametrization, one should first substitute $\beta \to \beta + \frac{1}{3}\alpha$, and then $\alpha \to 2\alpha$.)

$$
\begin{aligned}
0 \;=\; & \frac{\sqrt{AB}}{16r^2A^5B^4}\Big(16(A-1)A^2B^4(5\alpha+(\alpha-2\beta)A-14\beta)-32r(\alpha-4\beta)A^2B^3B' \\
& +4r^2B^2\Big(7(3\alpha-8\beta)B^2A'^2+2AB\left(2(8\beta-3\alpha)BA''+(16\beta-5\alpha)A'B'\right) \\
& +A^2\left(8(\alpha-4\beta)BB''+(32\beta-11\alpha)B'^2\right)+4\gamma A^4B^2-4\gamma A^3B^2\Big) \\
& -4r^3B\Big(4ABB''\left((\alpha-4\beta)BA'+(\alpha-6\beta)AB'\right) \\
& +B'\left(-7(\alpha-4\beta)B^2A'^2+4AB\left((\alpha-4\beta)BA''+\beta A'B'\right)-(\alpha-8\beta)A^2B'^2+4\gamma A^3B^2\right)\Big) \\
& +r^4(\alpha-2\beta)\Big(6ABB'^2\left(A'B'-2AB''\right) \\
& +B^2\left(-8AA'B'B''+B'^2\left(7A'^2-4AA''\right)-4A^2B''^2\right)+7A^2B'^4\Big) \\
& +B^{(3)}\left(8r^4(\alpha-2\beta)A^2B^2B'+16r^3(\alpha-4\beta)A^2B^3\right)\Big)
\end{aligned}
$$

The second equation contains the third-order derivative $A^{(3)} = A'''$:

$$
\begin{aligned}
0 \;=\; & \frac{\sqrt{AB}}{2r^2A^5B^4\left(2(\alpha-4\beta)B+r(\alpha-2\beta)B'\right)^2}\Bigg(\\
& 16(\alpha-3\beta)(\alpha-4\beta)(A-1)A^3(\alpha+2\beta+(\alpha-2\beta)A)B^5 \\
& +16(\alpha-3\beta)A^2B^4\Big(2(\alpha-4\beta)(-2\alpha+2\beta+(\alpha-2\beta)A)BA' \\
& \quad +(\alpha-2\beta)(A-1)A(\alpha+2\beta+(\alpha-2\beta)A)B'\Big)r \\
& +4AB^3\Big(-4\beta(\alpha-4\beta)\gamma B^2A^4-4\left(-\beta(\alpha-4\beta)\gamma B^2+(\alpha-2\beta)^2(\alpha-3\beta)B'^2\right)A^3 \\
& \quad +(\alpha-3\beta)B'\left(4BA'(\alpha-2\beta)^2+\left(3\alpha^2-4\beta\alpha-16\beta^2\right)B'\right)A^2 \\
& \quad -2(\alpha-3\beta)B\left(\left(3\alpha^2-8\beta\alpha+8\beta^2\right)A'B'-2\alpha(\alpha-4\beta)BA''\right)A \\
& \quad -(\alpha-4\beta)(\alpha-3\beta)(5\alpha+8\beta)B^2A'^2\Big)r^2 \\
& +4(\alpha-4\beta)B^2\Big(+2(\alpha-2\beta)\gamma B^2B'A^5+2(\alpha-2\beta)\gamma B^2B'A^4 \\
& \quad +\left(-4\beta\gamma A'B^3+2(\alpha+4\beta)(\alpha-3\beta)B'B''B-(\alpha-3\beta)(3\alpha+4\beta)B'^3\right)A^3 \\
& \quad +2(\alpha-3\beta)B\left(\alpha BB'A''+A'\left(\alpha BB''-2(\alpha+\beta)B'^2\right)\right)A^2 \\
& \quad -\alpha(\alpha-3\beta)B^2A'\left(5A'B'+26BA''\right)A+28\alpha(\alpha-3\beta)B^3A'^3\Big)r^3 \\
& +(\alpha-2\beta)B\Big(-4\gamma B^2\left((\alpha-6\beta)B'^2-2(\alpha-4\beta)BB''\right)A^4 \\
& \quad +\Big(-(\alpha-3\beta)(5\alpha+4\beta)B'^4+8\alpha(\alpha-3\beta)BB''B'^2-4(\alpha-2\beta)\gamma B^3A'B' \\
& \quad -4(\alpha-4\beta)(\alpha-3\beta)B^2B''^2\Big)A^3 \\
& \quad +2(\alpha-3\beta)BB'\left(4\alpha BB'A''+A'\left((4\beta-5\alpha)B'^2+2(3\alpha-8\beta)BB''\right)\right)A^2 \\
& \quad -(\alpha-3\beta)B^2A'B'\left(3(7\alpha-4\beta)A'B'+52\alpha BA''\right)A+56\alpha(\alpha-3\beta)B^3A'^3B'\Big)r^4 \\
& -(\alpha-2\beta)^2AB'\left(AB'^2+B\left(A'B'-2AB''\right)\right)\Big(+2\gamma A^2B^2+(\alpha-3\beta)A'B'B \\
& \quad +(\alpha-3\beta)A\left(B'^2-2BB''\right)\Big)r^5 \\
& +\left(16r^3\alpha(\alpha-3\beta)(\alpha-4\beta)A^2B^5+8r^4\alpha(\alpha-2\beta)(\alpha-3\beta)A^2B'B^4\right)A^{(3)}\Bigg)
\end{aligned}
$$

Separation of modes in the linearized theory

Solving the full nonlinear field equations is clearly a challenge. One can make initial progress by restricting the metric to infinitesimal fluctuations about flat space, defining $h_{\mu\nu} = \kappa^{-1}(g_{\mu\nu} - \eta_{\mu\nu})$ and then restricting attention to field equations linearized in $h_{\mu\nu}$, or equivalently by restricting attention to quadratic terms in $h_{\mu\nu}$ in the action.

The action then becomes

$$
\begin{aligned}
I_{\text{Lin}} &= \int d^4x \{ -\frac{1}{4} h^{\mu\nu}(2\alpha\Box - \gamma)\Box P^{(2)}_{\mu\nu\rho\sigma} h^{\rho\sigma} \\
&\qquad + \frac{1}{2} h^{\mu\nu}[6\beta\Box - \gamma]\Box P^{(0;\mathrm{s})}_{\mu\nu\rho\sigma} h^{\rho\sigma} \} \; ;
\end{aligned}
$$

$$
\begin{aligned}
P^{(2)}_{\mu\nu\rho\sigma} &= \frac{1}{2}(\theta_{\mu\rho}\theta_{\nu\sigma} + \theta_{\mu\sigma}\theta_{\nu\rho}) - P^{(0;\mathrm{s})}_{\mu\nu\rho\sigma} \\
P^{(0;\mathrm{s})}_{\mu\nu\rho\sigma} &= \frac{1}{3}\theta_{\mu\nu}\theta_{\rho\sigma} \qquad \theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu} \qquad \omega_{\mu\nu} = \partial_\mu\partial_\nu/\Box \; ,
\end{aligned}
$$

where the indices are lowered and raised with the background metric $\eta_{\mu\nu}$.

From this linearized action one deduces the dynamical content of the linearized theory: *positive-energy* massless spin-two, *negative-energy* massive spin-two with mass $m_2 = \gamma^{\frac{1}{2}}(2\alpha)^{-\frac{1}{2}}$ and *positive-energy* massive spin-zero with mass $m_0 = \gamma^{\frac{1}{2}}(6\beta)^{-\frac{1}{2}}$.

K.S.S. 1978

$\sim\sim\sim$

A simple model of what has happened can be made with a single scalar field and a higher-derivative action coupled to a source J:

$$I_{\rm hd} = \int d^4x(-\tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \tfrac{1}{2}\alpha\partial_\mu\phi\Box\partial^\mu\phi + J\phi)$$

Going over to momentum space k^μ, one can solve for ϕ and then separate the propagator into partial fractions:

$$\phi = \frac{J/\alpha}{k^2(k^2+1/\alpha)} = \frac{J}{k^2} - \frac{J}{k^2+1/\alpha}$$

similar to the structure found in quadratic gravity, but without the spin complications.

For this scalar model system, one can introduce interpolating fields ψ and λ giving the same dynamics:

$$I_{\text{interpolating}} = \int d^4x(-\tfrac{1}{2}\partial_\mu\psi\partial^\mu\psi + \tfrac{1}{2}\partial_\mu\lambda\partial^\mu\lambda + \frac{1}{2\alpha}\lambda^2 + J(\psi+\lambda))$$

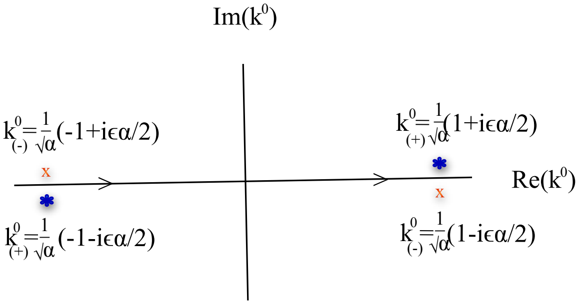
Noting the opposite signs for the ψ and λ kinetic terms in the Lagrangian, one finds for λ a momentum-space massive propagator with $m_\lambda^2 = 1/\alpha$ but with a nonstandard sign of the residue:

$$\lambda = \frac{-J}{k^2 + 1/\alpha}$$

The interpretation of this nonstandard sign for the massive interpolating field λ presents a devil's alternative. Supply a $\pm i\epsilon$ shift in the denominator in order to make the inverse Fourier transform unambiguous,

$$\lambda = \frac{-J}{k^2 + 1/\alpha \pm i\epsilon}$$

The $\pm i\epsilon$ sign choice with the negative-residue propagator for λ amounts, in the Feynman-ordered inverse Fourier transform, to a choice between negative-energy states with positive norm in the state vector space, or positive-energy states with negative norm in the state vector space (aka "ghosts"). It is such ghosts that one hopes might be avoided by a careful treatment in the asymptotic safety program.



Inverse-Fourier transform $\pm i\epsilon$ choices for a negative-residue propagator

Static and spherically symmetric solutions

Now we come to the question of what happens to spherically symmetric gravitational solutions in the higher-curvature theory. Work in Schwarzschild coordinates

$$ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

In the linearized theory, one then finds the general solution to the source-free field equations $H^L_{\mu\nu} = 0$, where $C, C^{2,0}, C^{2,+}, C^{2,-}, C^{0,+}, C^{0,-}$ are integration constants:

$$A(r) =$$
$$1 - \frac{C^{20}}{r} - C^{2+}\frac{e^{m_2 r}}{2r} - C^{2-}\frac{e^{-m_2 r}}{2r} + C^{0+}\frac{e^{m_0 r}}{r} + C^{0-}\frac{e^{-m_0 r}}{r}$$
$$+\frac{1}{2}C^{2+}m_2 e^{m_2 r} - \frac{1}{2}C^{2-}m_2 e^{-m_2 r} - C^{0+}m_0 e^{m_0 r} + C^{0-}m_0 e^{-m_0 r}$$
$$B(r) =$$
$$C + \frac{C^{20}}{r} + C^{2+}\frac{e^{m_2 r}}{r} + C^{2-}\frac{e^{-m_2 r}}{r} + C^{0+}\frac{e^{m_0 r}}{r} + C^{0-}\frac{e^{-m_0 r}}{r}$$

- As one might expect from the dynamics of the linearized theory, the general static, spherically symmetric solution is a combination of a massless Newtonian $1/r$ potential plus rising and falling Yukawa potentials arising in both the spin-two and spin-zero sectors.
- When coupling to non-gravitational matter fields is made *via* standard $h^{\mu\nu}T_{\mu\nu}$ minimal coupling, one gets values for the integration constants from the specific form of the source stress tensor. Requiring asymptotic flatness and coupling to a point-source positive-energy matter delta function $T_{\mu\nu} = \delta^0_\mu \delta^0_\nu M \delta^3(\vec{x})$, for example, one finds

$$\begin{aligned} A(r) &= 1 + \frac{\kappa^2 M}{8\pi\gamma r} - \frac{\kappa^2 M(1+m_2 r)}{12\pi\gamma}\frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M(1+m_0 r)}{48\pi\gamma}\frac{e^{-m_0 r}}{r} \\ B(r) &= 1 - \frac{\kappa^2 M}{8\pi\gamma r} + \frac{\kappa^2 M}{6\pi\gamma}\frac{e^{-m_2 r}}{r} - \frac{\kappa^2 M}{24\pi\gamma}\frac{e^{-m_0 r}}{r} \end{aligned}$$

  with specific combinations of the Newtonian $1/r$ and falling Yukawa potential corrections arising from the spin-two and spin-zero sectors.

Note that in the Einstein-plus-quadratic-curvature theory, there is no Birkhoff theorem. For example, in the linearized theory, coupling to the stress tensor for an extended source like a perfect fluid with pressure P constrained within a radius ℓ by an elastic membrane,

$$T_{\mu\nu} = \mathrm{diag}[P, [P - \tfrac{1}{2}\ell\delta(r-\ell)]r^2, [P - \tfrac{1}{2}\ell\delta(r-\ell)]r^2 \sin^2\theta, 3M(4\pi\ell^3)^{-1}]\,,$$

one finds for the external $B(r)$ function

$$\begin{aligned} B(r) = 1 - \frac{\kappa^2 M}{8\pi\gamma r} + \frac{\kappa^2 e^{-m_2 r}}{\gamma r}\left\{\frac{M}{2\pi\ell^3}\left[\frac{\ell\cosh(m_2\ell)}{m_2^2} - \frac{\sinh(m_2\ell)}{m_2^3}\right]\right. \\ \left. -P\left[\frac{\sinh(m_2\ell)}{m_2^3} - \frac{\ell\cosh(m_2\ell)}{m_2^2} + \frac{\ell^2\sinh(m_2\ell)}{3m_2}\right]\right\} \\ -\frac{\kappa^2 e^{-m_0 r}}{2\gamma r}\left\{\frac{M}{4\pi\ell^3}\left[\frac{\ell\cosh(m_0\ell)}{m_0^2} - \frac{\sinh(m_0\ell)}{m_0^3}\right]\right. \\ \left. -P\left[\frac{\sinh(m_0\ell)}{m_0^3} - \frac{\ell\cosh(m_0\ell)}{m_0^2} + \frac{\ell^2\sinh(m_0\ell)}{3m_0}\right]\right\} \end{aligned}$$

which limits to the point-source result as $\ell \to 0$.

Frobenius Asymptotic Analysis

Asymptotic analysis of the field equations near the origin leads to study of the *indicial equations* for behavior as $r \to 0$. K.S.S. 1978
Let

$$\begin{aligned} A(r) &= a_s r^s + a_{s+1} r^{s+1} + a_{s+2} r^{s+2} + \cdots \\ B(r) &= b_t r^t + b_{t+1} r^{t+1} + b_{t+2} r^{t+2} + \cdots \end{aligned}$$

and analyze the conditions necessary for the lowest-order terms in r of the field equations $H_{\mu\nu} = 0$ to be satisfied. This gives the following results, for the general $\alpha,\ \beta$ theory:

$$\begin{aligned} (s,t) &= (1,-1) \quad \text{with 4 free parameters} \\ (s,t) &= (0,0) \quad \text{with 3 free parameters} \\ (s,t) &= (2,2) \quad \text{with 6 free parameters} \end{aligned}$$

Lü, Perkins, Pope & K.S.S., in preparation

Now suppose one puts an "egg-shell" δ-function source at some small distance ϵ from the origin. Consider solving these sourced equations, similarly to the linearized theory analysis. Inside the shell, the solution can only be of the (0,0) nonsingular type, which needs no source. Suppose that outside one has a solution that would be of (2,2) type if one continued it all the way in to $r = 0$.

Count parameters: 3 inside + 6 outside = 9 initially. However, there are 6 continuity and 'jump' conditions coming from the field equations. So one really has 9-6=3 parameters still free. These 3 so-far unfixed parameters are just what is needed for 2 boundary conditions at infinity, to eliminate the rising exponential solutions, plus the 'trivial' parameter that is fixed by requiring $g_{00} \to -1$ as $r \to \infty$.

Conclusion: the exterior (2,2) solution works well with a source in the full non-linear theory. Exterior (1,-1) and (0,0) solutions would, however, be *overdetermined*. So coupling to a standard positive-energy source works only in the (2,2) family.

No-hair Theorems and Horizons

W. Nelson, Phys.Rev. D82 (2010) 104026; arXiv:1010.3986; H. Lü, A. Perkins, C.N. Pope & K.S.S., in preparation

- For $\beta > 0$ (*i.e.* for non-tachyonic $m_0^2 > 0$), take the trace of the $H_{\mu\nu} = 0$ field equation: $\left(\Box - \frac{\gamma}{6\beta}\right) R = 0$. Then multiply by $\lambda^{\frac{1}{2}} R$ and integrate with $\int \sqrt{h}$ over a 3D spatial slice at a fixed time, on which h_{ab} is the 3D metric and $\lambda = -t^a t^b g_{ab}$ is the norm of the timelike Killing vector t^a orthogonal to the slice. Integrating by parts, one obtains

$$\int d^3x\sqrt{h}[D^a(\lambda^{\frac{1}{2}} R D_a R) - \lambda^{\frac{1}{2}}(D^a R)(D_a R) - m_0^2 \lambda^{\frac{1}{2}} R^2] = 0$$

  where D_a is a 3D covariant derivative on the spatial slice.

  From this, provided the boundary term arising from the total derivative gives a zero contribution, and for $m_0^2 > 0$, one learns $R = 0$. The boundary at spatial infinity gives a vanishing contribution provided $R \to 0$ as $r \to \infty$.
- The inner boundary at a horizon null-surface will give a zero contribution since $\lambda = 0$ there.

Consequently, for asymptotically flat solutions with a horizon, one concludes that one must have $R = 0$. This already excludes the possibility of the scalar m_0 Yukawa correction found in the limit as $r \to \infty$. So, for solutions that do have such a scalar Yukawa correction to the classic GR behavior, one directly concludes: *there can be no horizon.*

What about the non-trace part of the field equation and the spin-two m_2 Yukawa corrections? Nelson's paper would have allowed one to make a similar conclusion for the rest of $R_{\mu\nu}$. Unfortunately, detailed analysis of his paper shows that it has a fundamental flaw: instead of a sum of squares of the same sign, one gets squares of opposite signs. Lü, Perkins, Pope & K.S.S., in preparation

If one assumes the existence of a horizon, one can carry out a similar expansion and parameter count to the ones made near the origin. Assuming asymptotic flatness at infinity, one obtains $R = 0$ as above. The field equations then become identical to those in the special $\beta = 0$ case, *i.e.* with just a $(\text{Weyl})^2$ term and no R^2 term in the action.

Counting parameters in the expansion around the horizon, subject to the $R = 0$ condition, one finds 3 parameters. This is the same count as in the (1,-1) family of the expansion around the origin when subjected to the $R = 0$ condition. So asymptotically flat solutions with a horizon must belong uniquely to the (1-1) family, which contains the Schwarzschild solution itself. The Schwarzschild solution is characterized by two parameters: the mass M of the black hole, plus the trivial g_{00} normalization at infinity. So in the higher-derivative theory, there is just one "non-Schwarzschild" (1,-1) parameter.

Away from Schwarzschild in the (1,-1) family

Lü, Perkins, Pope & K.S.S., in preparation

Considering variation of this "non-Schwarzschild" parameter away from the Schwarzschild value, it is clear that changing it has to do something to the solution at infinity. For a solution assumed to have a horizon, and holding $R = 0$, the only thing that can happen initially is that the rising exponential is turned on, *i.e.* asymptotic flatness is lost. So, for asymptotically flat solutions with a horizon *in the vicinity of the Schwarzschild solution*, the only spherically symmetric static solution is Schwarzschild itself.

This conclusion is formalized by considering infinitesimal variations of a solution away from Schwarzschild and proving a no-hair theorem for the *linearized* equation in the variation. This can successfully be done for coefficients α that are not too large (*i.e.* for spin-two masses m_2 that are not too small). One concludes that the Schwarzschild black hole is at least in general *isolated* as an asymptotically flat solution with a horizon.

Non-Schwarzschild Black Holes

Lü, Perkins, Pope & K.S.S., PRL 114, 171601 (2015); arXiv 1502.01028

Now the question arises what happens when one moves a finite distance away from Schwarzschild in terms of the (1,-1) non-Schwarzschild parameter. Does the loss of asymptotic flatness persist, or does something else happen, with solutions arising that cannot be treated by a linearized analysis in deviation from Schwarzchild?

This can only be answered numerically. In consequence of the trace no-hair theorem, the assumption of a horizon together with asymptotic flatness requires $R = 0$ for the solution, so the calculations can effectively be done in the $R - C^2$ theory with $\beta = 0$, in which the field equations thankfully can be reduced to a system of two second-order equations.

The study of non-Schwarzschild solutions is more easily carried out with a metric parametrization

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,,$$

i.e. by letting $A(r) = 1/f(r)$.

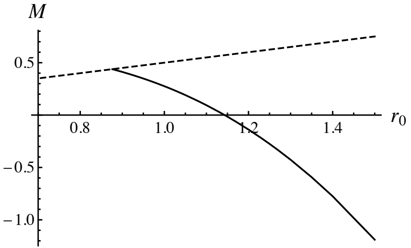
For $B(r)$ vanishing linearly in $r - r_0$ for some r_0, analysis of the field equations shows that one must then also have $f(r)$ similarly linearly vanishing at r_0, and accordingly one has a horizon. One can thus make near-horizon expansions

$$\begin{aligned} B(r) &= c\left[(r-r_0) + h_2\,(r-r_0)^2 + h_3\,(r-r_0)^3 + \cdots\right] \\ f(r) &= f_1\,(r-r_0) + f_2\,(r-r_0)^2 + f_3\,(r-r_0)^3 + \cdots \end{aligned}$$

and the parameters h_i and f_i for $i \geq 2$ can then be solved-for in terms of r_0 and f_1. For the Schwarzschild solution, one has $f_1 = 1/r_0$, so it is convenient to parametrize the deviation from Schwarzschild using a non-Schwarzschild parameter δ with

$$f_1 = \frac{1+\delta}{r_0}\,.$$

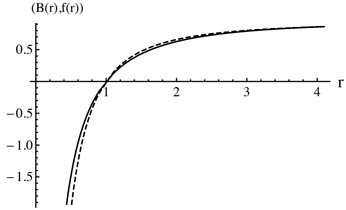
The task then becomes that of finding values of $\delta \neq 0$ for which the generic rising exponential behavior as $r \to \infty$ is suppressed. What one finds is that there do indeed exist asymptotically flat non-Schwarzschild black holes provided the horizon radius r_0 exceeds a certain minimum value $r_0^{\rm min}$. For $\alpha = \frac{1}{2}$, one finds the following phases of black holes:



Black-hole masses as a function of horizon radius r_0, with a branch point at $r_0^{\rm min} \simeq 0.876$; the dashed line denotes Schwarzschild black holes and the solid line denotes non-Schwarzschild black holes.

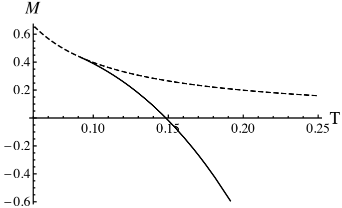
Properties of the non-Schwarzschild black holes

One can see from the mass M versus horizon radius r_0 that there is a maximum mass $M^{\rm max} = \frac{1}{2} r_0^{\rm min} > 0$ for the non-Schwarzschild branch of black holes. The non-Schwarzschild black hole is found to have a spin-two falling g_{00} Yukawa term $-\frac{c^{2-}}{r} e^{-m_2 r}$ with a coefficient c^{2-} that is of the *same* sign as M. This sign is *opposite* to that expected from the linearized theory's coupling to a standard positive-energy shell source. Otherwise, the solution extending from the origin out to spatial infinity looks generally similar to the Schwarzschild black hole and belongs to the (1,-1) solution class.



Non-Schwarzschild black hole for $M \sim .276$ with a horizon at $r = 1$. The dashed line denotes $B(r)$ and the solid line denotes $f(r) = 1/A(r)$.

Note that the mass M of the non-Schwarzschild black hole *decreases* as r_0 increases. Consequently, there is a horizon radius $r_0^{\mathrm{m}=0} \simeq 1.143$ at which it becomes massless. The relation between the mass M and the Hawking temperature T is shown by

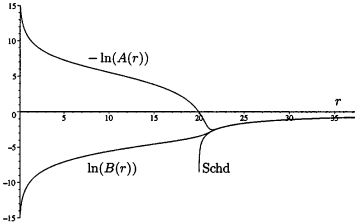


Non-Schwarzschild black hole mass M as function of temperature T. The dashed line denotes Schwarzschild black holes and the solid line denotes non-Schwarzschild black holes

The specific heat $C = \partial M/\partial T$ is negative for both Schwarzschild and non-Schwarzschild black holes. At a given temperature T, C is more negative for the non-Schwarzschild black hole.

(2,2) solutions without horizons

For asymptotically flat solutions with spin-two Yukawa coefficient $c^{2-} < 0$, *i.e.* of the same sign as that found in the linearized theory when coupled to positive-energy sources, one finds instead numerical solutions that can continue on in to mesh with the (2,2) family obtained from Frobenius asymptotic analysis around the origin. Such solutions have no horizon; they were investigated numerically in the theory with $m_2 = m_0$ (*i.e.* $\alpha = 3\beta$) by Bob Holdom. B. Holdom, Phys.Rev. D66 (2002) 084010; hep-th/0206219



Horizonless solution with $c_1 < 0$, behaving as r^2 in both $A(r)$ and $B(r)$ as $r \to 0$.

- *There is no horizon* in this set of minimally-coupled, Yukawa-corrected solutions. Solutions asymptotically approach the Schwarzschild solution for large r, but differ strikingly in what would have been the inner-horizon region.
- This is in accord with generic conclusions from the parameter count for solutions with horizons and from the linearized no-hair theorems. Generic asymptotically free solutions have to break free from the parameter-count restriction (three) for solutions with horizons, and need the full parameter set (six) found for (2,2) family solutions. For a generic $R - C^2 + R^2$ theory solution, there will be both spin-two and spin-zero falling Yukawa terms as one approaches spatial infinity. Together with the trivial time-rescaling parameter and the mass M, this makes four welcome parameters. One then needs two more solution parameters to ensure cancellation of the unwelcome rising spin-two and spin-zero exponential terms.

- Finding such horizon-free solutions is computationally intricate. The most secure way to find them is to use the shooting method coming in from large radii, started out using the linearized solutions at spatial infinity and then matching on to a shooting solution integrated outwards, started out using the (2,2) family series solution near the origin.
- Although there is a curvature singularity at the origin in the $(2,2)$ class of solutions (*e.g.* for this class, one has $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 20a_2^{-2}r^{-8} + \cdots$), this is a *timelike* singularity, unlike the *spacelike* singularity of the Schwarzschild solution.

Stability Issues

What does this all mean for "physical" black holes? The above static analysis does not yet consider the issue of stability, *i.e.* what happens to time-dependent solutions obtained from perturbations away from the static solutions. Since no closed-form version of the exact non-Schwarzschild (1,-1) or (2,2) solutions is available, this is not an easy question to address. However, one can get some information by considering the stability of the Schwarzschild solution itself within the higher-derivative theory.

- In the $R + R^2$ theory, study of the normal modes about the Schwarzschild solution shows it to be stable. This is perhaps not surprizing, since that theory is classically equivalent to ordinary Einstein gravity plus a scalar field with a peculiar potential, for which the ordinary GR stability considerations and no-hair theorem should apply. Whitt, Starobinsky

- When the $(\text{Weyl})^2$ term is present in the action, however, the stability situation is different: there may be a phase structure, depending on the value of $\mu = \frac{m_2 M}{M_{\rm Pl}^2}$, where m_2 is the spin-two particle mass, M is the mass of the black hole and $M_{\rm Pl}$ is the Planck mass. For $\mu \gg 1$, *i.e.* "largeish" black holes, one obtains stability for the Schwarzschild solution. For $\mu \leq 1$, on the other hand, stability is not guaranteed.
- This was studied by Brian Whitt Phys. Rev. D32 (1985) 379 , who showed that the $R + (\text{Weyl})^2$ theory should be stable for $\mu \geq 0.44$ but raised the question of whether an instability could set in for $\mu < 0.44$. Indeed, he suggested that there could be a bifurcation of the spherically symmetric solution set into two branches at this value.

- Whitt's detailed calculation seemed to show, nonetheless, that there was still no instability, at least in a static perturbation analysis (*i.e.* for $k = 0$ momentum modes).
- This analysis has, however, been challenged in a paper by Y.S. Myung Phys.Rev. D88 (2013) 2, 024039; arxiv:1306.3725 who argues that Whitt did not do the Schwarzschild stability analysis properly and instead does find, from a nonstatic $k \neq 0$ analysis, an instability of the Schwarzschild solution for $\mu < \mathcal{O}(1)$. This is similar to Schwarzschild instabilities found in massive gravity theories Babichev & Fabbri; Brito, Cardoso & Pani

This raises the possibility of a phase structure for black hole solutions in higher-derivative gravity. The entropy of a non-Schwarzschild black hole of a given mass turns out to be less than that of the Schwarzschild black hole of the same mass. So could there be a hierarchy of stabilities involving the Schwarzschild and non-Schwarzschild black holes and the (2,2) solution with a naked singularity?

$R + R^2$ theory

The massive spin-two ghost can be eliminated at the classical level by setting $\alpha \to 0_+$, for which $m_2 \to \infty$. Choosing $\beta > 0$ makes the spin-zero mode non-tachyonic, and the resulting $\int d^4x\sqrt{-g}(-R + \beta R^2)$ theory is equivalent to GR coupled to a *non-ghost* scalar field. This remains true at the full nonlinear level B. Whitt 1984 , with an action (including also a cosmological term)

$$
\begin{aligned}
I_{R+\text{spin zero}} &= \int d^4x\sqrt{-g}(-R + \beta R^2 - 2\Lambda) \\
&\leftrightarrow \int d^4x\sqrt{-g}(-R \\
&\quad -6\beta^2(1+2\beta\phi)^{-2}(\nabla_\mu\phi\nabla^\mu\phi + \frac{1}{6\beta}\phi^2 + \frac{1}{3\beta^2}\Lambda))
\end{aligned}
$$

- One can redefine the scalar field $\phi = (e^{\tilde{\phi}/\sqrt{3}} - 1)/2\beta$ in order to produce a scalar Lagrangian with a canonical kinetic term and a transformed potential $-\frac{1}{2}\nabla_\mu\tilde{\phi}\nabla^\mu\tilde{\phi} - V(\tilde{\phi})$, where

$$V(\tilde{\phi}) = \frac{1}{4\beta}(1 - e^{-\tilde{\phi}/\sqrt{3}})^2 + 2\Lambda e^{-2\tilde{\phi}/\sqrt{3}}$$

- It is thus clear that, for large $\tilde{\phi}$, the potential $V(\tilde{\phi})$ becomes very flat. This was the reason for the attractiveness (at times) of the $\int d^4x\sqrt{-g}(-R + \beta R^2)$ theory for inflation purposes.

  A.A. Starobinsky 1980; Mukhanov & Chibisov 1981

  The coefficient β sets the scale for the potential. Restoring a $1/\kappa^2$ coefficient for the Einstein-Hilbert action $\int\sqrt{-g}R$, the mass of the scalar mode is $m_0^2 = (6\kappa^2\beta)^{-1}$; applications for inflation typically take this mass scale to be something like 10^{-6} of the Planck scale.